MATH 33A: LINEAR ALGEBRA AND APPLICATIONS
SPRING 2017 - LECTURE 3
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MIDTERM 2

Your Name

Your Student ID number

Your TA Section

By signing below, you confirm that you did not cheat on this exam. No exam booklet without a signature will be graded.

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INSTRUCTIONS

- Please do not open this booklet until you are told to do so.
- You are only to use items necessary for writing. No other devices of any kind are permitted.
- No books or notes.
- If you have a question at any time during the exam, please raise your hand.
- You will receive points only for work written on the numbered pages. Please use the reverse side as scratch paper.
- Make sure to write legibly. Illegible work will not be graded.
- Make sure to show all your work and justify your answers fully.
- If you finish early, please wait in your seat until the time is called.

SCORE

1. 10
2. 10
3. 10
4. 9
5. 10

TOTAL 49

1. *a*) (6 pts) Find a basis for the image of the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{pmatrix}.$$

$$\begin{bmatrix} 5 \\ 15 \\ 5 \\ 16 \end{bmatrix} - \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 1 \\ 1 \end{bmatrix}$$

b) (4 pts) Find a basis for the kernel of the matrix A from part *a*).

6a
$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix} \begin{matrix} \\ -3(I) \\ -(I) \\ -2(I) \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -3 & 0 \end{bmatrix} \begin{matrix} -3(III) \\ \\ \\ -3(III) \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} \begin{matrix} \\ \\ \\ \div -3 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} +2(IV) \\ \\ -(IV) \\ \\ \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \text{Swap}(II, III) \\ \text{Swap}(III, IV) \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{basis}(\text{im}(A)) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix} \right\}$$

1b
$$\begin{bmatrix} 2 \\ 6 \\ 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}$$

$$2 \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 6 \\ 1 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix}$$

$$5 \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

$$\text{basis}(\ker(A)) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

2. *a*) (4 pts) Let $\mathfrak{B}$ be the standard basis of $\mathbb{R}^2$. Describe geometrically the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose $\mathfrak{B}$-matrix is

$$A = \begin{pmatrix} -0.28 & 0.96 \\ 0.96 & 0.28 \end{pmatrix}.$$

(Hint: $(0.96)^2 + (0.28)^2 = 1$.)

b) (6 pts) Write down a basis $\mathfrak{B}'$ for $\mathbb{R}^2$ such that the $\mathfrak{B}'$-matrix of T is diagonal, and write down the $\mathfrak{B}'$-matrix of T.

2a

$2v_2^2 - 1 = 0.28$ $\quad$ $2v_1^2 - 1 = -0.28$

$2v_2^2 = 1.28$ $\quad$ $2v_1^2 = 0.72$

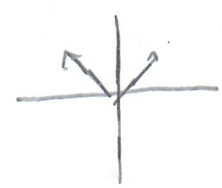
$v_2^2 = 0.64$ $\quad$ $v_1^2 = 0.36$

$v_2 = 0.8$ $\quad$ $v_1 = 0.6$

Geometrically, the linear transformation represented by A is the reflection across the line parallel to the vector $\begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$

b

$\mathfrak{B}' = \left\{ \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}, \begin{bmatrix} -0.8 \\ 0.6 \end{bmatrix} \right\}$



$\mathfrak{B}'$ matrix: $[A]_{\mathfrak{B}'} = B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

3. Let $\mathfrak{B}$ be the standard basis of $\mathbb{R}^3$, and let

$$\mathfrak{B}' = \left\{ \vec{v}_1' = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \vec{v}_2' = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}, \vec{v}_3' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

where the vectors $\vec{v}_1'$, $\vec{v}_2'$, and $\vec{v}_3'$ have been written in terms of $\mathfrak{B}$. The set $\mathfrak{B}'$ is also a basis of $\mathbb{R}^3$. Let

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

be the $\mathfrak{B}$-matrix of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$.

a) (2 pts) Write down the change of basis matrix ${}_{\mathfrak{B}}S_{\mathfrak{B}'}$.

b) (4 pts) Write down the change of basis matrix ${}_{\mathfrak{B}'}S_{\mathfrak{B}}$.

c) (4 pts) Find the $\mathfrak{B}'$-matrix B of T.

3a

$${}_{\mathfrak{B}}S_{\mathfrak{B}'} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b Let us note that ${}_{\mathfrak{B}}S_{\mathfrak{B}'}$ is orthogonal, thus ${}_{\mathfrak{B}'}S_{\mathfrak{B}} = {}_{\mathfrak{B}}S_{\mathfrak{B}'}{}^T$

$${}_{\mathfrak{B}'}S_{\mathfrak{B}} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$-2/\sqrt{2}$

c

$${}_{\mathfrak{B}}A_{\mathfrak{B}}\,{}_{\mathfrak{B}}S_{\mathfrak{B}'} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\sqrt{2} & -1 \\ \sqrt{2} & 0 & 1 \\ 0 & -\sqrt{2} & -1 \end{bmatrix}$$

$$B = {}_{\mathfrak{B}'}S_{\mathfrak{B}}\,{}_{\mathfrak{B}}A_{\mathfrak{B}}\,{}_{\mathfrak{B}}S_{\mathfrak{B}'} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\sqrt{2} & -1 \\ \sqrt{2} & 0 & 1 \\ 0 & -\sqrt{2} & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & -\sqrt{2} \\ 0 & -\sqrt{2} & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & -\sqrt{2} \\ 0 & -\sqrt{2} & -1 \end{bmatrix}$$

4. (10 pts) Let $V \subseteq \mathbb{R}^n$ be a linear subspace. Show that the function $\text{proj}_V : \mathbb{R}^n \to V$ defined by

$$\text{proj}_V(\vec{x}) = \vec{x}^{\parallel}$$

is a linear transformation.

(Hint: Do *not* attempt to write down a matrix for proj_V.)

In order for $\text{proj}_V(\vec{x})$ to be a linear transformation, we must prove that

1. $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

2. $T(k\vec{x}) = kT(\vec{x})$

where T is the transformation $\text{proj}_V(\vec{x})$.

T can be represented as $(\vec{v_1}\cdot\vec{x})\vec{v_1} + \ldots + (\vec{v_n}\cdot\vec{x})\vec{v_n}$, where the vectors $\{\vec{v_1}, \ldots, \vec{v_n}\}$ are an orthonormal basis of V. Below are the proofs for 1, 2.

1) $T(\vec{x}+\vec{y}) = (\vec{v_1}\cdot(\vec{x}+\vec{y}))\vec{v_1} + \ldots + (\vec{v_n}\cdot(\vec{x}+\vec{y}))\vec{v_n} = (\vec{v_1}\cdot\vec{x} + \vec{v_1}\cdot\vec{y})\vec{v_1} + \ldots + (\vec{v_n}\cdot\vec{x} + \vec{v_n}\cdot\vec{y})\vec{v_n}$

$= (\vec{v_1}\cdot\vec{x})\vec{v_1} + \ldots + (\vec{v_n}\cdot\vec{x})\vec{v_n} + (\vec{v_1}\cdot\vec{y})\vec{v_1} + \ldots + (\vec{v_n}\cdot\vec{y})v_n = T(\vec{x}) + T(\vec{y})$

by the properties of the dot product.

2) $T(k\vec{x}) = (\vec{v_1}\cdot(k\vec{x}))\vec{v_1} + \ldots + (\vec{v_n}\cdot(k\vec{x}))\vec{v_n} = (k(\vec{v_1}\cdot\vec{x}))\vec{v_1} + \ldots + (k(\vec{v_n}\cdot\vec{x}))\vec{v_n}$

$= k[(\vec{v_1}\cdot\vec{x})\vec{v_1} + \ldots + (\vec{v_n}\cdot\vec{x})\vec{v_n}] = kT(\vec{x})$

Thus by proving 1, 2, we conclude that $\text{proj}_V(\vec{x})$ is indeed a linear transformation. ■

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5. Let P be the following subset of $\mathbb{R}^3$:

$$P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x + y + z = 0 \right\}.$$

$\begin{bmatrix} -r \end{bmatrix} \begin{bmatrix} \end{bmatrix} \begin{bmatrix} -r-s \\ r \\ s \end{bmatrix}$

$= \text{span } \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

a) (2 pts) Show that P is a linear subspace of $\mathbb{R}^3$.

b) (8 pts) Find an orthonormal basis for the subspace P from part *a*). Make sure to show all your work.

5a

P is a linear subspace because the matrix representing P is a kernel.

P can be represented by $A = [1, 1, 1]$. Thus the transformation $A\vec{x}$

can be represented $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. By the definition of P, $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

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meaning that P is the kernel of matrix $[1 \ 1 \ 1]$ which has 3 columns, so the kernel of P is a subspace of $\mathbb{R}^3$

b $\quad x = -y - z$

basis (kernel (P)) = $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$v_1 \qquad v_2$

$\|\vec{v}_1\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\vec{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$

$\vec{u}_1 \cdot \vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1/\sqrt{2}$

$(\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$

$\vec{v}_2^{\perp} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}$

$\|\vec{v}_2^{\perp}\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \frac{\sqrt{3}}{\sqrt{2}}$

$u_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ -\sqrt{2} \end{bmatrix}$

basis = $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ -\sqrt{2} \end{bmatrix} \right\}$

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