MATH 33A: LINEAR ALGEBRA AND APPLICATIONS
SPRING 2017 - LECTURE 3
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MIDTERM 2

Your Name

Your Student ID number

Your TA Section

By signing below, you confirm that you did not cheat on this exam. No exam booklet without a signature will be graded.

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INSTRUCTIONS

- Please do not open this booklet until you are told to do so.
- You are only to use items necessary for writing. No other devices of any kind are permitted.
- No books or notes.
- If you have a question at any time during the exam, please raise your hand.
- You will receive points only for work written on the numbered pages. Please use the reverse side as scratch paper.
- Make sure to write legibly. Illegible work will not be graded.
- Make sure to show all your work and justify your answers fully.
- If you finish early, please wait in your seat until the time is called.

SCORE

1. ______
2. ______
3. ______
4. ______
5. ______

TOTAL ______

1. *a*) (6 pts) Find a basis for the image of the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{pmatrix}.$$

Solution. By row reduction to the RREF (or by inspection), you will find that the first, third, and fifth column are non-redundant, and so the image of A has a basis

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 4 \\ 9 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \\ 2 \end{pmatrix} \right\}.$$

b) (4 pts) Find a basis for the kernel of the matrix A from part *a*).

Solution. By writing the second and fourth columns (which are redundant) as linear combinations of the preceding (non-redundant) columns, you will find that the kernel of A has a basis

$$\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

2. *a*) (4 pts) Let $\mathfrak{B}$ be the standard basis of $\mathbb{R}^2$. Describe geometrically the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose $\mathfrak{B}$-matrix is

$$A = \begin{pmatrix} -0.28 & 0.96 \\ 0.96 & 0.28 \end{pmatrix}.$$

(Hint: $(0.96)^2 + (0.28)^2 = 1$.)

Solution. You will recognize A as being of the form

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

with $a^2 + b^2 = 1$ and hence, by Lecture 6, A represents reflection about the line parallel to

$$\begin{pmatrix} a+1 \\ b \end{pmatrix} = \begin{pmatrix} 0.72 \\ 0.96 \end{pmatrix},$$

which is parallel to

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

b) (6 pts) Write down a basis $\mathfrak{B}'$ for $\mathbb{R}^2$ such that the $\mathfrak{B}'$-matrix of T is diagonal, and write down the $\mathfrak{B}'$-matrix of T.

Solution.

$$\mathfrak{B}' = \left\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -4 \\ 3 \end{pmatrix} \right\}.$$

$${}_{\mathfrak{B}'}[T]_{\mathfrak{B}'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3. Let $\mathfrak{B}$ be the standard basis of $\mathbb{R}^3$, and let

$$\mathfrak{B}' = \left\{ \vec{v}_1' = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \ \vec{v}_2' = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}, \ \vec{v}_3' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

where the vectors $\vec{v}_1'$, $\vec{v}_2'$, and $\vec{v}_3'$ have been written in terms of $\mathfrak{B}$. The set $\mathfrak{B}'$ is also a basis of $\mathbb{R}^3$. Let

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

be the $\mathfrak{B}$-matrix of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$.

a) (2 pts) Write down the change of basis matrix ${}_{\mathfrak{B}}S_{\mathfrak{B}'}$.

Solution.

$${}_{\mathfrak{B}}S_{\mathfrak{B}'} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

b) (4 pts) Write down the change of basis matrix ${}_{\mathfrak{B}'}S_{\mathfrak{B}}$.

Solution. This is the inverse of the matrix from *a*), and since that matrix is orthogonal, its inverse is just its transpose:

$${}_{\mathfrak{B}'}S_{\mathfrak{B}} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is actually the matrix itself, as the matrix in question is symmetric.

c) (4 pts) Find the $\mathfrak{B}'$-matrix B of T.

Solution. This is a multiplication you're just going to have to carry out carefully.

$$B = {}_{\mathfrak{B}'}S_{\mathfrak{B}}\,{}_{\mathfrak{B}}[T]_{\mathfrak{B}}\,{}_{\mathfrak{B}}S_{\mathfrak{B}'} = {}_{\mathfrak{B}'}S_{\mathfrak{B}}\,A\,{}_{\mathfrak{B}}S_{\mathfrak{B}'} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & -\sqrt{2} \\ 0 & -\sqrt{2} & -1 \end{pmatrix}.$$

4. (10 pts) Let $V \subseteq \mathbb{R}^n$ be a linear subspace. Show that the function $\text{proj}_V : \mathbb{R}^n \to V$ defined by

$$\text{proj}_V(\vec{x}) = \vec{x}^{\|}$$

is a linear transformation.

(Hint: Do *not* attempt to write down a matrix for proj_V.)

Solution. Recall that any vector space V has a basis, and by using the Gram-Schmidt process, any basis can be converted into an orhonormal basis.

Let $\{\vec{u}_1, ..., \vec{u}_m\}$ be an orthonormal basis for V.

We know that the projection onto V can be written as

$$\text{proj}_V(\vec{x}) = (\vec{x} \bullet \vec{u}_1)\vec{u}_1 + ... + (\vec{x} \bullet \vec{u}_m)\vec{u}_m.$$

By using this expression, we can check that the two definitive properties of linearity are satisfied:

1) $\text{proj}_V(\vec{x} + \vec{y}) = ((\vec{x} + \vec{y}) \bullet \vec{u}_1)\vec{u}_1 + ... + ((\vec{x} + \vec{y}) \bullet \vec{u}_m)\vec{u}_m$

$= ((\vec{x} \bullet \vec{u}_1) + (\vec{y} \bullet \vec{u}_1))\vec{u}_1 + ... + ((\vec{x} \bullet \vec{u}_m) + (\vec{y} \bullet \vec{u}_m))\vec{u}_m$

$= (\vec{x} \bullet \vec{u}_1)\vec{u}_1 + (\vec{y} \bullet \vec{u}_1)\vec{u}_1 + ... + (\vec{x} \bullet \vec{u}_m)\vec{u}_m + (\vec{y} \bullet \vec{u}_m)\vec{u}_m$

$= (\vec{x} \bullet \vec{u}_1) + ... + (\vec{x} \bullet \vec{u}_m)\vec{u}_m + (\vec{y} \bullet \vec{u}_1) + ... + (\vec{y} \bullet \vec{u}_m)\vec{u}_m$

$= \text{proj}_V(\vec{x}) + \text{proj}_V(\vec{y}).$

2) $\text{proj}_V(k\vec{x}) = (k\vec{x} \bullet \vec{u}_1)\vec{u}_1 + ... + (k\vec{x} \bullet \vec{u}_m)\vec{u}_m$

$= k(\vec{x} \bullet \vec{u}_1)\vec{u}_1 + ... + k(\vec{x} \bullet \vec{u}_m)\vec{u}_m$

$= k((\vec{x} \bullet \vec{u}_1)\vec{u}_1 + ... + (\vec{x} \bullet \vec{u}_m)\vec{u}_m)$

$= k(\text{proj}_V(\vec{x})).$

5. Let P be the following subset of $\mathbb{R}^3$:

$$P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x + y + z = 0 \right\}.$$

a) (2 pts) Show that P is a linear subspace of $\mathbb{R}^3$.

Solution. From Math 32A, we know that P is a plane through the origin defined by $x + y + z = 0$. Hence, P is a subspace.

b) (8 pts) Find an orthonormal basis for the subspace P from part *a*). Make sure to show all your work.

Solution. This is again a two-step process. First, you can pick any two non-parallel vectors that are orthogonal to the normal vector of P, namely $\langle 1, 1, 1 \rangle$. For example, you can start with

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

This is a basis for P. Now you can apply the Gram-Schmidt process to obtain an orthonormal basis of P:

$$\left\{ \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2}\sqrt{3} \\ 1/\sqrt{2}\sqrt{3} \\ -\sqrt{2}/\sqrt{3} \end{pmatrix} \right\}.$$