MATH 33A: LINEAR ALGEBRA AND APPLICATIONS
FALL 2016 - LECTURE 2
Jukka Keranen

MIDTERM 2

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By signing below, you confirm that you did not cheat on this exam. No exam booklet without a signature will be graded.

Dgl Wg
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INSTRUCTIONS

- Please do not open this booklet until you are told to do so.
- You are only to use items necessary for writing. No other devices of any kind are permitted.
- No books or notes.
- If you have a question at any time during the exam, please raise your hand.
- You will receive points only for work written on the numbered pages. Please use the reverse side as scratch paper.
- Make sure to write legibly. Illegible work will not be graded.
- Make sure to show all your work and justify your answers fully.
- If you finish early, please wait in your seat until the time is called.

SCORE

1. 10
2. 7
3. 8
4. 8
5. 2

TOTAL 35

1. *a*) (6 pts) Find a basis for the image of the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{pmatrix}.$$

b) (4 pts) Find a basis for the kernel of the matrix A from part *a*).

$$\begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{pmatrix} \xrightarrow{\substack{-3(\mathrm{I}) \\ +(\mathrm{II})}} \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 4 & -12 & -4 \\ 0 & 0 & 0 & 3 \end{pmatrix} \xrightarrow{\substack{-4(\mathrm{II}) \\ \div 3}} \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3 pivots, so 3 vectors form basis

Basis of img(A): $\boxed{\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ -6 \\ 3 \end{pmatrix} \right\}}$

$x + 2z + 4t = 0 \rightarrow x = -2z$

$y - 3z - t = 0 \rightarrow y = 3z \rightarrow \begin{pmatrix} -2n \\ 3n \\ n \\ 0 \end{pmatrix}$

$t = 0$

Let $n = 1$

Basis of ker(A): $\boxed{\left\{ \begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\}}$

$$0.04\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}$$

2. *a*) (2 pts) Let $\mathfrak{B}$ be the standard basis of $\mathbb{R}^2$. Describe geometrically the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose $\mathfrak{B}$-matrix is

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ a+b \end{pmatrix}$$

$$A = \begin{pmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{pmatrix}.$$

$\begin{matrix} 3 & 2 \\ 4 & 3 \end{matrix}$ $\quad 0.64 - 0.48 = 0.16$

b) (4 pts) Write down a basis $\mathfrak{B}'$ for $\mathbb{R}^2$ such that the $\mathfrak{B}'$-matrix of T is diagonal, and write down the $\mathfrak{B}'$-matrix of T.

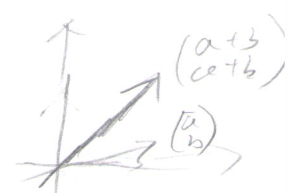
c) (2 pts) Describe geometrically some linear transformation $R : \mathbb{R}^2 \to \mathbb{R}^2$ such that there is *no* basis $\mathfrak{B}'$ of $\mathbb{R}^2$ with respect to which the $\mathfrak{B}'$-matrix of R is diagonal.

d) (2 pts) Write down the $\mathfrak{B}$-matrix of the transformation R you described in part *c*), where $\mathfrak{B}$ is the standard basis of $\mathbb{R}^2$.

a) The linear transformation is the projection onto the line spanned by vector $\begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$ (b/c it is OTF $\begin{pmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{pmatrix}$ where $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a unit vector)

b)

c) A collapse of a vector onto the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$



Why? 1/2

d) $\boxed{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}}$

8

3. Let $\mathfrak{B}$ be the standard basis of $\mathbb{R}^3$, and let

$$\mathfrak{B}' = \left\{ \vec{v}_1' = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix},\ \vec{v}_2' = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix},\ \vec{v}_3' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

where the vectors $\vec{v}_1'$, $\vec{v}_2'$, and $\vec{v}_3'$ have been written in terms of $\mathfrak{B}$. The set $\mathfrak{B}'$ is also a basis of $\mathbb{R}^3$. Let

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

be the $\mathfrak{B}$-matrix of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$.

a) (2 pts) Write down the change of basis matrix ${}_{\mathfrak{B}}S_{\mathfrak{B}'}$.

b) (4 pts) Write down the change of basis matrix ${}_{\mathfrak{B}'}S_{\mathfrak{B}}$.

c) (4 pts) Find the $\mathfrak{B}'$-matrix B of T.

√ +2

$${}_BS_{B'} = \boxed{\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

√ +2

$$\left(\begin{array}{ccc|ccc} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\substack{-\text{II} \\ +\text{I}}} \left(\begin{array}{ccc|ccc} 0 & -\sqrt{2} & 0 & 1 & -1 & 0 \\ \sqrt{2} & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\substack{\div -\sqrt{2} \\ \div \sqrt{2}}} \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

b/c ${}_{B'}S_B = {}_BS_{B'}^{-1}$

$$\boxed{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

+1

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 1 & 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

√ +2

$$[T]_{B'} = {}_{B'}S_B\,[T]_B\,{}_BS_{B'}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

+1

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}}$$

$\begin{matrix}1 & 3\\ 4 & 12\end{matrix}$ $\begin{matrix}1 & 4\\ 3 & 12\end{matrix}$

4. *a)* (4 pts) State the Rank-Nullity Theorem.

For the remainder of Problem 4., let A be an $n \times m$ matrix, and let B be an invertible $m \times m$ matrix.

b) (2 pts) Show that $Im(A) = Im(AB)$.

c) (2 pts) Show that $\dim(\ker(A)) = \dim(\ker(AB))$.

d) (2 pts) Determine the rank of $B^T A^T$ in terms of the rank of A.

$\begin{matrix}1 & 3\\ 2 & 7\end{matrix}$ img: $\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}3\\7\end{pmatrix}$

$\begin{matrix}1 & 3\\ 2 & 7\end{matrix}$ img $\begin{pmatrix}1\\3\end{pmatrix}\begin{pmatrix}2\\7\end{pmatrix}$

a) Nullity(A) + Rk(A) = m (for an n×m mx) 4.

b) ~~Null(A) + Rk(A) = m and Null(AB) + Rk(AB) = m (b/c AB is also n×m)~~

~~Null(A) + Rk(A) = Null~~ 2.

~~A~~ B is invertible, then its RREF is I_m and its img can be described as span{e_1 ... e_m} ∴ its span is $\mathbb{R}^m$. B/c of this, applying A onto B would be like applying A onto any value in $\mathbb{R}^m$ so img(A) = img(AB)

c) Null(A) + Rk(A) = m, Null(AB) + Rk(AB) = m (b/c AB is n×m too)

Null(A) + Rk(A) = Null(AB) + Rk(AB) 2.

Rk(A) = Rk(AB) b/c Rk is equivalent to dim(img(M)) for some mx M and b/c img(A) = img(AB), dim(img(A)) = dim(img(AB))

so

Null(A) + $\cancel{Rk(A)}$ = Null(AB) + $\cancel{Rk(A)}$

Null(A) = Null(AB) → $\boxed{\dim(\ker(A)) = \dim(\ker(AB))}$

d) B^t does not affect rank(A) b/c it is invertible

But dim(ker(A)) = dim(ker($B^T A^T$)) so 0.

2

5. Let V be the following subset of $\mathbb{R}^4$:

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid x+y+z+w=0,\ \ x-y+z+w=0 \right\}.$$

+0

a) (2 pts) Show that the set V above is a subspace of $\mathbb{R}^4$ by showing that V is the kernel of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$.

b) (8 pts) Find an orthonormal basis for the subspace V above. Make sure to show all your work.

b) $V = \left\{ \cancel{\begin{pmatrix} -a-b-c \\ a \\ b \\ c \end{pmatrix}, \begin{pmatrix} a-3-c \\ a \\ b \\ c \end{pmatrix}} \right\}$

let $a,b,c = 1$ $\left\{ \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\cancel{u_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{12}} \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}}$

$u_2 = \frac{1}{\|v_2\|} v_2 = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$$\boxed{\begin{aligned} u_1 &= \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ u_2 &= \frac{1}{2\sqrt{21}} \begin{pmatrix} 9 \\ -1 \\ -1 \\ -1 \end{pmatrix} \end{aligned}}$$

$v_1^\perp = v_1 - (u_2 \cdot v_1) u_2$ +2

$= \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \left(\frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right) \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{4}(6) \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -3/2 \\ 3/2 \\ 3/2 \\ 3/2 \end{pmatrix} = \begin{pmatrix} 9/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{pmatrix}$

$81 + 1 + 1 + 1 = 84$

$u_1 = \frac{1}{\|v_1^\perp\|} v_1^\perp = \frac{1}{\frac{\sqrt{84}}{2}} \begin{pmatrix} 9/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{pmatrix} = \frac{1}{\sqrt{21}} \cdot \frac{1}{2} \begin{pmatrix} 9 \\ -1 \\ -1 \\ -1 \end{pmatrix}$