MATH 33A: LINEAR ALGEBRA AND APPLICATIONS
SPRING 2017 - LECTURE 3
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MIDTERM 1 SOLUTIONS

Your Name

Your Student ID number

Your TA Section

By signing below, you confirm that you did not cheat on this exam. No exam booklet without a signature will be graded.

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INSTRUCTIONS

- Please do not open this booklet until you are told to do so.
- You are only to use items necessary for writing. No other devices of any kind are permitted.
- No books or notes.
- If you have a question at any time during the exam, please raise your hand.
- You will receive points only for work written on the numbered pages. Please use the reverse side as scratch paper.
- Make sure to write legibly. Illegible work will not be graded.
- Make sure to show all your work and justify your answers fully.
- If you finish early, please wait in your seat until the time is called.

SCORE

1. ______

2. ______

3. ______

4. ______

5. ______

TOTAL ______

1. *a*) (6 pts) Find the inverse of the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{pmatrix}.$$

b) (4 pts) Solve the system $A\vec{x} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$.

Solution.

a) You just form the "double-augmented matrix" and perform row reduction. You will find that the inverse of A is

$$A^{-1} = \begin{pmatrix} 3/2 & -1 & 1/2 \\ 1/2 & 0 & -1/2 \\ -3/2 & 1 & 1/2 \end{pmatrix}.$$

b) Simply multiply both sides of the equation by A^{-1} and use part *a*):

$$A\vec{x} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$\Rightarrow \vec{x} = A^{-1}\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3/2 & -1 & 1/2 \\ 1/2 & 0 & -1/2 \\ -3/2 & 1 & 1/2 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

REMARK This is the fundamental application of matrix inverses. If you solved part *b*) in any other way, you may have missed one of the basic points of this course. If you have any questions regarding this problem, please come and talk to me as soon as possible, and certainly before Midterm 2.

2. *a*) (6 pts) Find the matrix representing the following linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$:

1) reflection about the line $y = 2x$, followed by

2) counterclockwise rotation through $\pi/4$ radians, followed by

3) projection onto the y-axis.

b) (4 pts) Prove that the linear transformation T in part *a*) is not invertible.

Solution.

a) 1) The line $y = 2x$ is parallel to the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Hence, by using the template from Lecture 6, the reflection about the line $y = 2x$ is represented by

$$A = \begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix}.$$

2) The counterclockwise rotation through $\pi/4$ radians is represented by

$$B = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

3) The projection onto the y-axis is represented by

$$C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, the matrix M representing the composite transformation T is

$$M = CBA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1/5\sqrt{2} & 7/5\sqrt{2} \end{pmatrix}.$$

b) Any linear transformation is invertible if and only if the matrix representing it is invertible, and we already know that a 2×2 matrix M is invertible if and only if the determinant of M is non-zero. Further, we already know how actually to compute the determinant of a 2×2 matrix:

$$\det \begin{pmatrix} 0 & 0 \\ 1/5\sqrt{2} & 7/5\sqrt{2} \end{pmatrix} = (0)(7/5\sqrt{2}) - (0)(1/5\sqrt{2}) = 0.$$

Thus, T is not invertible.

3. (10 pts) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Show that the inverse T^{-1} of T is a linear transformation.

Solution. 0. By definition, T^{-1} is a linear transformation if

1) $T^{-1}(\vec{x} + \vec{y}) = T^{-1}(\vec{x}) + T^{-1}(\vec{y})$,

and

2) $T^{-1}(k\vec{x}) = kT^{-1}(\vec{x})$

for all $\vec{x}$, $\vec{y}$ in $\mathbb{R}^n$ and for all k in $\mathbb{R}$. Thus, we need to show that T^{-1} has both of these properties.

1. Since any invertible function $\mathbb{R}^n \to \mathbb{R}^n$ is onto, we know that for any $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$, there exist $\vec{x}'$ and $\vec{y}'$ in $\mathbb{R}^n$ such that

$$\vec{x} = T(\vec{x}') \text{ and } \vec{y} = T(\vec{y}').$$

2. Thus, we have for any $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ that

$$T^{-1}(\vec{x} + \vec{y}) = T^{-1}(T(\vec{x}') + T(\vec{y}')) = T^{-1}(T(\vec{x}' + \vec{y}')) = \vec{x}' + \vec{y}' = T^{-1}(\vec{x}) + T^{-1}(\vec{y}),$$

where the second equality holds because T is by assumption linear, and the final equality holds by the choice of $\vec{x}'$ and $\vec{y}'$ in Step 1. Thus, the property 1) from Step 0. holds.

3. Similarly, we have for any $\vec{x}$ in $\mathbb{R}^n$ and any k in $\mathbb{R}$ that

$$T^{-1}(k\vec{x}) = T^{-1}(kT(\vec{x}')) = T^{-1}(T(k\vec{x}')) = k\vec{x}' = kT^{-1}(\vec{x}),$$

where the second equality again holds because T is by assumption linear, and the final equality holds by the choice of $\vec{x}'$ in Step 1. Thus, the property 2) from Step 0. holds.

4. By Steps 2. and 3., we have shown that T^{-1} is a linear transformation.

REMARK You cannot appeal to the existence of the matrix A^{-1} in proving that T^{-1} is linear. Yes, we have a theorem which says that a function $\mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation if and only if it is represented by some matrix. However, what you are trying to prove here is precisely the statement that T^{-1} is a linear transformation. And so, at the beginning of the proof, you cannot yet assume that there exists a matrix A^{-1} representing T^{-1}. To make that assumption would be circular reasoning – in other words, that would be to assume the conclusion you are trying to prove. In fact, given an invertible linear transformation T, we defined A^{-1} to be the matrix that represents the unique linear transformation T^{-1}; so, our very definition of A^{-1} presupposes that we know T^{-1} is linear when T is linear (please see the beginning of Lecture 9).

4. *a*) (6 pts) Write down a numerical 2×2 matrix A such that

$$A^2 = A$$

and *all entries of A are nonzero.*

b) (4 pts) Write down a numerical 2×2 matrix B such that

$$B^3 = B$$

and *all entries of B are nonzero.*

Solution. For both *a*) and *b*), you can take the matrix to be

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

a) The point of this problem was to think geometrically. You know the list of those five fundamental linear transformations from Lecture 6. Which one of them could have the property required in *a*)? By considering the options, you should be able to see that a projection onto any line other than the coordinate axes will work. Indeed, if $\vec{x}$ is any vector in $\mathbb{R}^2$, and if A represents the projection onto any line in $\mathbb{R}^2$, we have

$$A^2(\vec{x}) = A(A(\vec{x})) = A(\vec{x}).$$

(Think about it!) In particular, you can take A to be the matrix representing projection onto the line $y = x$, which is the matrix above. The only reason you cannot take as your line either one of the coordinate axes is that the corresponding matrices would have zeroes in them.

b) The best way to solve *b*) is to realize that if A has the property in *a*), then

$$A^3 = A(A^2) = A(A) = A^2 = A.$$

Thus, you can take B to be the same matrix you chose as your A in *a*).

5. (10 pts) *Without using determinants*, find the values of the constant k such that the following matrix is invertible:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{pmatrix}.$$

Make sure to justify your answer fully.

Solution. 0. You know that a 3×3 matrix A is invertible if and only if the rank of A is 3. Further, by definition, the rank of A is the number of pivots in $RREF(A)$. Thus, all you need to do is to determine how the number of pivots in $RREF(A)$ depends on the value of k.

1. By performing elementary row operations, you can quickly put the given matrix in the form

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & k-1 \\ 0 & 0 & k^2 - 3k + 2 \end{pmatrix}.$$

From this, you can already see that there will be a pivot on the first two rows, no matter what the value of k is.

2. Further, there will be a pivot on the third row if and only if

$$k^2 - 3k + 2 \neq 0.$$

Thus, the only values of k for which the given matrix is not invertible are the two roots of the quadratic equation

$$k^2 - 3k + 2 = 0,$$

namely

$$k = 1 \text{ and } k = 2.$$

3. In sum, the given matrix is invertible for all values of the constant k other than 1 and 2.

REMARK It is unnecessary to work with the"double-augmented matrix" in this problem. It is also unnecessary to row reduce all the way to the $RREF(A)$, since the matrix shown in Step 1. already tells you everything you need to know.