MATH 33A: LINEAR ALGEBRA AND APPLICATIONS
FALL 2016 - LECTURE 2
Jukka Keranen

MIDTERM 1

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By signing below, you confirm that you did not cheat on this exam. No exam booklet without a signature will be graded.

Doug Wong
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INSTRUCTIONS

- Please do not open this booklet until you are told to do so.
- You are only to use items necessary for writing. No other devices of any kind are permitted.
- No books or notes.
- If you have a question at any time during the exam, please raise your hand.
- You will receive points only for work written on the numbered pages. Please use the reverse side as scratch paper.
- Make sure to write legibly. Illegible work will not be graded.
- Make sure to show all your work and justify your answers fully.
- If you finish early, please wait in your seat until the time is called.

SCORE

1. 10
2. 10
3. 8 → 10.
4. 8
5. 7 → 9

TOTAL 43 → 45. → 47

$$\begin{pmatrix} 1 & 3 & -3 \\ -2 & 3 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 3 & 6 \\ 1 & 3 & 5 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 \\ \\ \end{pmatrix}$$

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$$\begin{pmatrix} 1 & -3 & 3 \\ -2 & 3 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 3 & 6 \\ 1 & 3 & 5 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \\ \end{pmatrix}$$

1. *a*) (6 pts) Find the inverse of the following matrix:

$$A = \begin{pmatrix} 1 & 3 & 6 \\ 1 & 3 & 5 \\ 1 & 2 & 3 \end{pmatrix}.$$

b) (4 pts) Solve the system $A\vec{x} = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 6 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-\text{II}]{-\text{II}} \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & 3 & 5 & 0 & 1 & 0 \\ 0 & -1 & -2 & 0 & -1 & 1 \end{array}\right) \xrightarrow[+2\text{I}]{-3\text{III}} \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 4 & -3 \\ 0 & -1 & 0 & 2 & -3 & 1 \end{array}\right)$$

$$\xrightarrow[\div -1]{+\text{I}} \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 3 & -3 \\ 0 & 1 & 0 & -2 & 3 & -1 \end{array}\right) \xrightarrow{\text{rearrange}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & -3 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 6 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-\text{III}]{-\text{II}} \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & -1 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-2\text{II}]{-2\text{I}} \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 1 & 0 & -1 & 0 & -2 & 3 \end{array}\right)$$

$$\xrightarrow{+\text{I}} \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 1 & 0 & 0 & 1 & -3 & 3 \end{array}\right) \xrightarrow{\text{rearrange}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 3 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right)$$

$$\boxed{A^{-1} = \begin{pmatrix} 1 & -3 & 3 \\ -2 & 3 & -1 \\ 1 & -1 & 0 \end{pmatrix}}$$

✓ +6

$$Ax = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} \to A^{-1}Ax = A^{-1}\begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} \to x = A^{-1}\begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -3 & 3 \\ -2 & 3 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$$

✓ +4

$$= \begin{pmatrix} 4-15+9 \\ -8+15-3 \\ 4-5+0 \end{pmatrix} = \boxed{\begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}}$$

2. a) (6 pts) Find the matrix representing the following linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

1) counterclockwise rotation through $\pi/4$ radians, followed by

2) scaling by a factor of 2, followed by

3) reflection about the line $y = x$.

b) (4 pts) State whether this transformation is invertible. Justify your answer. If it is, find the matrix representing its inverse.

1) $S = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \rightarrow \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

2) $T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ 3) $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ flips x and y terms

Composed together $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix} = \boxed{\begin{pmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}}$

Yes there exists an inverse because we can sequentially undo all the linear transformations in the composition by multiplying with it's inverse

composition: UTS

inverse composition $= S^{-1}T^{-1}U^{-1}$

$\cos\frac{-\pi}{4} = \frac{\sqrt{2}}{2}$

$\sin\frac{-\pi}{4} = -\frac{\sqrt{2}}{2}$.

$$\begin{pmatrix} \cos\frac{-\pi}{4} & -\sin\frac{-\pi}{4} \\ \sin\frac{-\pi}{4} & \cos\frac{-\pi}{4} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} = \boxed{\begin{pmatrix} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \end{pmatrix}}$$

3. a) (4 pts) Write down a matrix that represents the projection onto the line through $\vec{0}$ and parallel to a unit vector $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

b) (4 pts) By using your answer to a), show that projections are not invertible.

c) (2 pts) What is the rank of the matrix in your answer to a)? Justify your answer.

a) $\boxed{\begin{pmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{pmatrix}}$

b) inverse $= \frac{1}{\det(mx)}\begin{pmatrix} u_2^2 & -u_1u_2 \\ -u_1u_2 & u_1^2 \end{pmatrix} = \frac{1}{u_1^2u_2^2 - (u_1u_2)(u_1u_2)}\begin{pmatrix} u_2^2 & -u_1u_2 \\ -u_1u_2 & u_2^2 \end{pmatrix}$

$= \frac{1}{0}\begin{pmatrix} u_2^2 & -u_1u_2 \\ -u_1u_2 & u_2^2 \end{pmatrix}$

2/4

$\boxed{\text{determinant} = 0 \therefore \text{inverse DNE}}$

c) Rank = 1 b/c if it were rank = 2, then it would have an inverse b/c for an nxn matrix, being of rank n in RREF implies the matrix has an inverse. However, b/c it does not have an inverse we know it cannot be rank 2. Also, b/c it is not $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ it is not rank 0 $\therefore$ it must be rank 1

$\begin{pmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{pmatrix} \xrightarrow[-\frac{u_2}{u_1}(\mathrm{I})]{} \begin{pmatrix} u_1^2 & u_1u_2 \\ 0 & 0 \end{pmatrix}$

2/2

4. *a*) (4 pts) Explain what is meant by the statement that matrix multiplication is non-commutative.

b) (2 pts) Write down two matrices that do not commute.

c) (4 pts) *Without computing the products*, explain why the two matrices you wrote down in *b*) do not commute.

a) for an arbitrary $n \times n$ matrix A and $n \times n$ matrix B AB does not necessarily equal BA (although it is possible) $\rightarrow$ $AB = BA$ is not always true 4.

b) $\boxed{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}}$ 2.

c) The matrices represent liner transformations $\begin{cases} x_0 = x + 2y \\ y_0 = 3x + 4y \end{cases}$ and $\begin{cases} x_0 = 5x + 6y \\ y_0 = 7x + 8y \end{cases}$. Applying the first set to the second set and vice-versa is not commutative b/c the actions of distributing and adding does not commute. 2

$A(B+C)+D \neq$ any similar combination

$\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ $\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$ $\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$

5. *a*) (5 pts) Suppose that $A^3 = I_3$. What is the rank of A? Justify your answer.

b) (5 pts) Write down a 2×2 matrix A such that $A^2 = -I_2$. Show that your matrix has the desired property.

a) Rank of A is 3 b/c the rank of I_3 is 3.
If one imagines as 3 linear equations, if one ??? repeatedly applied a linear transformation onto a system of equation that is inconsistent, one could not get a consistent system. ∴ A must represent a consistent system and ∴ the rank of the $n \times n$ matrix A is n. $n=3$ b/c only way for size of A^3 to work out to the 3x3 identity mx is if $n=3$.) So rank$(A)=3$ 2

b) $\boxed{\begin{pmatrix}0&-1\\1&0\end{pmatrix}}$ $\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=-1\begin{pmatrix}1&0\\0&1\end{pmatrix}$

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