Math 33A, Sec 3
Linear Algebra and Applications
J. Madrid

Final Exam. Jun 08-09, 2020

Instructions: You have 24 **hours** to complete this exam, **from Monday Jun 08 at 8:00 am to Tuesday Jun 09 at 8:00 am, Pacific time**. There are eight problems, worth a total of 45 points. This test is OPEN book and OPEN notes. Calculators are allowed.

For full credit show all of your work legibly and **justify all your answers!** (except in problem 1 (True or false question)). Collaboration is NOT allowed.

Please write your solutions in white paper, take photos of your solution, put all of them in a single file and convert to pdf, then upload to Gradescope by the deadline Tuesday Jun 09 at 8:00 am. Make sure that your pdf file CONTAIN your name and UID number. You don't need to attach your scratch work. Please **circle or box your final answers**.

Important information you should include in your pdf file:

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Question	Points	Score
1	4	
2	5	
3	4	
4	7	
5	4	
6	8	
7	7	
8	6	
Total:	45	

Problem 1. *4pts.*

Indicate which of the following are true or false; no justification is required: *[1pt Each question]*

i.) A square matrix with distinct eigenvalues is orthogonally diagonalizable. $\boxed{F}$

ii.) The nullity of a square matrix coincides with the algebraic multiplicity of 0 as an eigenvalue $\boxed{F}$

iii.) The matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 1 & -1 \\ 0 & 3 & 2 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ is invertible. $\boxed{F}$

iv.) There is a nonzero symmetric matrix A such that $A^3 = 0$. $\boxed{F}$

i) $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$, $\lambda = 1, 2$, but A is not symmetric

ii) $\lambda = 0$, $\ker(A - \lambda I) = \ker(A)$; $E_0 = \ker(A)$; $\dim(E_0) = \text{nullity}(A)$

geometric multiplicity

iii) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 1 & -1 \\ 0 & 3 & 2 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{III + II - I} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

iv) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \to A^2 = [1$

$A^3 = SD^3S^{-1}$; $A^3 = 0$ only if $D = 0$, $\cancel{S = 0}$ or $\cancel{S^{-1} = 0}$

ii) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \lambda = 0$ $E_0 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

$\dim(E_0) = 1$, but algebraic multiplicity is 3

Problem 2. *5pts.*

Let A be the matrix of a counterclockwise rotation with an angle of $\pi/3$ in $\mathbb{R}^2$ and let B be the matrix of the reflection through the line passing by $\begin{bmatrix}1\\2\end{bmatrix}$ and the origin. Let $Tx = ABx$ for all $x \in \mathbb{R}^2$.

i.) (2 points) Find the angle between $T\begin{bmatrix}1\\1\end{bmatrix}$ and $T\begin{bmatrix}1\\0\end{bmatrix}$.

ii.) (3 point) Find the inverse of AB.

i) $A = \begin{bmatrix}\cos(\pi/3) & -\sin(\pi/3)\\ \sin(\pi/3) & \cos(\pi/3)\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$

$\text{ref}(\vec{x}) = \vec{x}_{\parallel} - \vec{x}_{\perp} = 2\,\text{proj}_L(\vec{x}) - \vec{x} = 2(\vec{x}\cdot\vec{u})\vec{u} - \vec{x}$

$\vec{u} = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}$

$\text{ref}_L\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = 2\left(\begin{bmatrix}\frac{1}{5}\\ \frac{2}{5}\end{bmatrix}\right) - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-\frac{3}{5}\\ \frac{4}{5}\end{bmatrix}$

$\text{ref}_L\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = 2\left(\begin{bmatrix}\frac{2}{5}\\ \frac{4}{5}\end{bmatrix}\right) - \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\frac{4}{5}\\ +\frac{3}{5}\end{bmatrix}$

$B = \begin{bmatrix}-\frac{3}{5} & \frac{4}{5}\\ \frac{4}{5} & \frac{3}{5}\end{bmatrix}$; Since A is a rotation, it doesn't change the angle between the vectors, so we can skip multiplying by A.

$B\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{5}\\ \frac{7}{5}\end{bmatrix}$; $B\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-\frac{3}{5}\\ \frac{4}{5}\end{bmatrix} \Rightarrow \begin{bmatrix}\frac{1}{5}\\ \frac{7}{5}\end{bmatrix}\cdot\begin{bmatrix}-\frac{3}{5}\\ \frac{4}{5}\end{bmatrix} = (\sqrt{2})(1)\cos\theta$

$\Rightarrow 1 = \sqrt{2}\cos\theta \Rightarrow \boxed{\theta = \frac{\pi}{4}}$

ii) $B = B^{-1}$, $A^{-1} = \begin{bmatrix}\cos(-\pi/3) & \sin(-\pi/3)\\ \sin(-\pi/3) & \cos(-\pi/3)\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$

$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix}-\frac{3}{5} & \frac{4}{5}\\ \frac{4}{5} & \frac{3}{5}\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix} = \begin{bmatrix}-\frac{3}{10} - \frac{4\sqrt{3}}{10} & -\frac{3\sqrt{3}}{10} + \frac{4}{10}\\ \frac{4}{10} - \frac{3\sqrt{3}}{10} & \frac{4\sqrt{3}}{10} + \frac{3}{10}\end{bmatrix}$

$= \boxed{\frac{1}{10}\begin{bmatrix}-(3+4\sqrt{3}) & 4-3\sqrt{3}\\ 4-3\sqrt{3} & 3+4\sqrt{3}\end{bmatrix}}$

Problem 3. *4pts.*

Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \\ 1 & 5 \end{bmatrix}$.

i.) (2 points) Find an orthonormal basis for $Im(A)$.

i.) (2 points) Find the QR factorization of A.

i) $\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$, $\vec{u}_1 \cdot \vec{v}_2 = \frac{9}{3} = 3$; $\vec{v}_{2\perp} = \vec{v}_2 - 3\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

$\Rightarrow \vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$, $\vec{u}_2 = \frac{1}{\sqrt{18}} \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$

Orthonormal Basis for $Im(A) = \boxed{\left\{ \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \frac{1}{\sqrt{18}} \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} \right\}}$

ii) $Q = \begin{bmatrix} 2/3 & \frac{-1}{\sqrt{18}} \\ 2/3 & -1/\sqrt{18} \\ 1/3 & 4/\sqrt{18} \end{bmatrix}$

$R = \begin{bmatrix} \|\vec{v}_1\| & \vec{u}_1 \cdot \vec{v}_2 \\ 0 & \|\vec{v}_2^{\perp}\| \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{18} \end{bmatrix}$

$$\boxed{\begin{bmatrix} 2 & 1 \\ 2 & 1 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/\sqrt{18} \\ 2/3 & -1/\sqrt{18} \\ 1/3 & 4/\sqrt{18} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{18} \end{bmatrix}}$$

Problem 4. *7pts.*

Consider the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

i.) (2.5 points) Prove that A is diagonalizable, and find a diagonal matrix D and an invertible matrix S such that $A = SDS^{-1}$.

ii.) (2 points) Find S^{-1}.

iii.) (2.5 points) Find a 3×3 matrix B such that $B^2 = A$.

i) A is upper triangular, and thus $\lambda = 3, 2, 1$. A is a 3×3 matrix, and has 3 distinct eigenvalues. This means A must have an eigenbasis, and is thus diagonalizable.

$$E_1 = \ker \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)$$

$$E_2 = \ker \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$$

$$E_3 = \ker \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$\boxed{D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}}$$

ii)

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-\mathrm{III}]{\mathrm{I}+\mathrm{II},\ -\mathrm{II}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\xrightarrow[\mathrm{I}+\mathrm{II}]{\mathrm{I}-\mathrm{III}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \Rightarrow \boxed{S^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}}$$

iii) SEE NEXT PAGE

iii) $B^2 = A = SDS^{-1}$

KNOW: $(SDS^{-1})^t = SD^tS^{-1}$

$$B = (SDS^{-1})^{1/2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \left(\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & \sqrt{3} & \sqrt{3} \\ 0 & -\sqrt{2} & -\sqrt{2} \\ 0 & 0 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} \sqrt{3} & \sqrt{3}-\sqrt{2} & \sqrt{3}-\sqrt{2} \\ 0 & \sqrt{2} & \sqrt{2}+1 \\ 0 & 0 & 1 \end{bmatrix}}$$

Problem 5. *4pts.*

Consider the matrix $A = \begin{bmatrix} 2 & -1 & 2 & -6 \\ 1 & -1 & 2 & -3 \\ 3 & -1 & 2 & -9 \\ 1 & 0 & 0 & -3 \end{bmatrix}$. Find the orthogonal projection of $v = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$ onto $ker(A)$.

$$A \to \begin{array}{c} \mathrm{IV} \\ \\ \\ \mathrm{I} \end{array}\begin{bmatrix} 1 & 0 & 0 & -3 \\ 1 & -1 & 2 & -3 \\ 3 & -1 & 2 & -9 \\ 2 & -1 & 2 & -6 \end{bmatrix} \xrightarrow{\substack{\mathrm{I}-\mathrm{II} \\ 3\mathrm{I}-\mathrm{III} \\ 2\mathrm{I}-\mathrm{IV}}} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{\substack{\mathrm{II}-\mathrm{III} \\ \mathrm{II}-\mathrm{IV}}} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{array}{l} x_1 = 3x_4 \\ x_2 = 2x_3 \end{array}$$

$$Ker(A) = span\left(\begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$\vec{u}_1 = \begin{bmatrix} 3/\sqrt{10} \\ 0 \\ 0 \\ 1/\sqrt{10} \end{bmatrix}, \quad \vec{v}_2 \cdot \vec{u}_1 = 0 \to \vec{v}_2^{\perp} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}; \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$proj_{Ker(A)}(\vec{v}) = \left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3/\sqrt{10} \\ 0 \\ 0 \\ 1/\sqrt{10} \end{bmatrix}\right)\begin{bmatrix} 3/\sqrt{10} \\ 0 \\ 0 \\ 1/\sqrt{10} \end{bmatrix} + \left(\begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 3/2 \\ 2 \\ 1 \\ 1/2 \end{bmatrix}}$$

Problem 6. *8pts.*

Consider the quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$ defined by $q(x_1, x_2, x_3) = 3x_1^2 + 5x_3^2 + 4x_2^2 - 6x_1x_3 - 4x_1x_2$.

i.) (2 points) Find the associated symmetric matrix A, for which $q(x) = x \cdot Ax$ for all x in $\mathbb{R}^3$.

ii.) (3 points) Determine the definiteness of q.

iii.) (3 points) Observe that A is invertible. Consider the quadratic form defined by $q_2(x) := x \cdot A^{-1}x$. Determine the definitness of q_2.

i) $6x_1x_3 \to 3$ in position $(1,3), (3,1)$ $\quad -4x_1x_2 \to -2$ in position $(1,2), (2,1)$

$$\boxed{A = \begin{bmatrix} 3 & -2 & 3 \\ -2 & 4 & 0 \\ 3 & 0 & 5 \end{bmatrix}}$$

ii) $\det\begin{bmatrix} 3 \end{bmatrix} = 3 > 0 \quad \det\begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix} = 12 - 4 = 8 > 0$

$$\det\begin{bmatrix} 3 & -2 & 3 \\ -2 & 4 & 0 \\ 3 & 0 & 5 \end{bmatrix} = (3)(4)(5) - (-2)(-2)(5) - (3)(4)(3)$$
$$= 60 - 20 - 36 = 4 > 0$$

$\therefore$ $\boxed{q \text{ is positive definite}}$

iii)
$$\left[\begin{array}{ccc|ccc} 3 & -2 & 3 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ 3 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{I}-\text{III}]{\frac{1}{3}\text{I},\ 2\text{I}+3\text{II}} \left[\begin{array}{ccc|ccc} 1 & -\frac{2}{3} & 1 & \frac{1}{3} & 0 & 0 \\ 0 & 8 & 6 & 2 & 3 & 0 \\ 0 & -2 & -2 & 1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{8}\text{II}} \left[\begin{array}{ccc|ccc} 1 & -\frac{2}{3} & 1 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 3/4 & \frac{1}{4} & 3/8 & 0 \\ 0 & -2 & -2 & 1 & 0 & -1 \end{array}\right] \xrightarrow[\text{III}+2\text{II}]{\text{I}+\frac{2}{3}\text{II}} \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1/2 & 1/4 & 0 \\ 0 & 1 & 3/4 & 1/4 & 3/8 & 0 \\ 0 & 0 & -1/2 & 3/2 & 3/4 & -1 \end{array}\right]$$

$$\xrightarrow[\substack{\text{I}+3/2\text{III} \\ -2\text{III}}]{\text{I}+3\text{III}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 5/2 & -3 \\ 0 & 1 & 0 & -5/2 & 3/2 & -3/2 \\ 0 & 0 & 1 & -3 & -3/2 & 2 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 5 & 5/2 & -3 \\ 5/2 & 3/2 & -3/2 \\ -3 & -3/2 & 2 \end{bmatrix}$$

$\det\begin{bmatrix} 5 \end{bmatrix} = 5 > 0$; $\det\begin{bmatrix} 5 & 5/2 \\ 5/2 & 3/2 \end{bmatrix} = \frac{5}{4} > 0$; $\det(A^{-1}) = \frac{1}{4} > 0$

$\therefore$ $\boxed{q_2 \text{ is Positive definate}}$

Problem 7. *7pts.*

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the refletion about the line spanned by $v = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$.

i.) (2 points) Find an orthonormal eigenbasis β for L.

ii.) (2 points) Find the β–matrix B of L (this is the matrix of L with respect to β).

iii.) (3 points) Find the matrix A of L with respect to the standard basis of $\mathbb{R}^3$.

i) $\lambda = -1, 1$

$E_1 = \text{span}\left(\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix}\right)$

E_{-1}: $3x_1 + 4x_3 = 0 \Rightarrow \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -3/4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

$= \text{span}\left(\begin{bmatrix} 4/5 \\ 0 \\ -3/5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

$$\boxed{\beta = \left\{\begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix}, \begin{bmatrix} 4/5 \\ 0 \\ -3/5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}}$$

ii) $$\boxed{B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}$$

iii) $A = SBS^{-1}$; $S = \begin{bmatrix} 3/5 & 4/5 & 0 \\ 0 & 0 & 1 \\ 4/5 & -3/5 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 3/5 & 4/5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 4/5 & -3/5 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{II}]{\substack{5/3\,\text{I} \\ 4\text{I} - 3\text{III}}} \left[\begin{array}{ccc|ccc} 1 & 4/3 & 0 & 5/3 & 0 & 0 \\ 0 & 5 & 0 & 4 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[1/5\,\text{II}]{\text{I} - 4/15\,\text{II}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/5 & 0 & 4/5 \\ 0 & 1 & 0 & 4/5 & 0 & -3/5 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$A = S\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 3/5 & 0 & 4/5 \\ 4/5 & 0 & -3/5 \\ 0 & 1 & 0 \end{bmatrix} = S\begin{bmatrix} 3/5 & 0 & 4/5 \\ -4/5 & 0 & 3/5 \\ 0 & -1 & 0 \end{bmatrix}$$

→ NEXT PAGE

$$\begin{bmatrix} 3/5 & 4/5 & 0 \\ 0 & 0 & 1 \\ 4/5 & -3/5 & 0 \end{bmatrix} \begin{bmatrix} 3/5 & 0 & 4/5 \\ -4/5 & 0 & 3/5 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{9-16}{25} & 0 & \frac{24}{25} \\ 0 & -1 & 0 \\ \frac{24}{25} & 0 & +\frac{7}{25} \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} -7/25 & 0 & 24/25 \\ 0 & -1 & 0 \\ 24/25 & 0 & 7/25 \end{bmatrix}}$$

Problem 8. *6pts.*

Find a Singular Value Decomposition for: $B = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$.

$B^T B = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$; $B = U \Sigma V^T$

$\det \begin{bmatrix} 4-\lambda & 6 \\ 6 & 13-\lambda \end{bmatrix} = (4-\lambda)(13-\lambda) - 36 = 52 - 17\lambda + \lambda^2 - 36$

$= \lambda^2 - 17\lambda + 16 = (\lambda - 16)(\lambda - 1)$

$\lambda = 16, 1 \Rightarrow \sigma = 4, 1$

$E_{16} = \ker \begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}\right)$

$E_1 = \ker \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}\right)$

$V = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$

$\vec{u}_1 = \frac{1}{4} B \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \frac{1}{4\sqrt{5}} \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{4\sqrt{5}} \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$\vec{u}_2 = \frac{1}{1} B \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ -2\sqrt{5} \end{bmatrix}$

$$\boxed{B = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}}$$