Problem 1 (6 points). Find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that the $\mathfrak{B}$-matrix of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \vec{x}$$

is

$$B = \begin{array}{cc} T(\vec{v}_1) & \\ \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} & \begin{matrix} \vec{v}_1 \\ \vec{v}_2 \end{matrix} \end{array}$$

or explain why one does not exist.

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$$

$$A\vec{v}_1 = 5\vec{v}_1 \qquad\qquad A\vec{v}_2 = -\vec{v}_2$$

$$\begin{bmatrix} a+2b \\ 4a+3b \end{bmatrix} = \begin{bmatrix} 5a \\ 5b \end{bmatrix} \qquad \begin{bmatrix} c+2d \\ 4c+3d \end{bmatrix} = \begin{bmatrix} -c \\ -d \end{bmatrix}$$

$$a+2b = 5a \qquad c+2d = -c$$

$$4a+3b = 5b \qquad 4c+3d = -d$$

$$-4a+2b = 0 \qquad 2c+2d = 0$$

$$4a-2b = 0 \qquad 4c+4d = 0$$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} \qquad \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -s \\ s \end{bmatrix}$$

$$= t\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \text{ for } t \in \mathbb{R} \qquad = s\begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ for } s \in \mathbb{R}$$

A basis $\mathfrak{B}$ can be $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ (choose $t=2$, $s=1$)

Problem 2 (6 points). Find a basis for the kernel of A, and determine the dimension of the kernel of A.

$$A = \begin{bmatrix} 1 & 1 & 5 & 1 \\ 0 & 1/2 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 1/3 & 2/3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3t \\ -2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{for } t \in \mathbb{R}$$

basis for $\ker(A) = \left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\dim(\ker(A)) = 1$

Problem 3 (6 points). Find an orthonormal basis of the image of the matrix

$$\begin{array}{c} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ -1 & 0 & -1 \end{bmatrix}. \\ \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \end{array}$$

$$\vec{u}_1 = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} -\frac{1}{\sqrt{21}} \\ \frac{4}{\sqrt{21}} \\ \frac{2}{\sqrt{21}} \end{bmatrix} \quad \vec{u}_3 = \begin{bmatrix} -\frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \\ -\frac{3}{\sqrt{14}} \end{bmatrix}$$

$$\|\vec{v}_1\| = \sqrt{4+1+1} = \sqrt{6}$$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \qquad \vec{u}_1 \cdot \vec{v}_2 = \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}} = \frac{4}{\sqrt{6}}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{4}{\sqrt{6}} \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{\cancel{8}4}{\cancel{6}3} \\ \frac{\cancel{4}2}{\cancel{6}3} \\ -\frac{\cancel{4}2}{\cancel{6}3} \end{bmatrix} = \begin{bmatrix} -1/3 \\ 4/3 \\ 2/3 \end{bmatrix}$$

$$\|\vec{v}_2^{\perp}\| = \sqrt{\frac{1}{9} + \frac{16}{9} + \frac{4}{9}} = \sqrt{\frac{21}{9}} = \frac{\sqrt{21}}{3}$$

$$\begin{array}{r} 14 \\ \times 14 \\ \hline 56 \\ 14 \\ \hline 196 \end{array}$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{3}{\sqrt{21}} \begin{bmatrix} -1/3 \\ 4/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{21}} \\ \frac{4}{\sqrt{21}} \\ \frac{2}{\sqrt{21}} \end{bmatrix}$$

$$\vec{u}_1 \cdot \vec{v}_3 = \frac{4}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} = \frac{7}{\sqrt{6}} \qquad \vec{u}_2 \cdot \vec{v}_3 = -\frac{2}{\sqrt{21}} + \frac{8}{\sqrt{21}} - \frac{2}{\sqrt{21}} = \frac{4}{\sqrt{21}}$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2$$

$$= \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} - \frac{7}{\sqrt{6}} \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -1/\sqrt{6} \end{bmatrix} - \frac{4}{\sqrt{21}} \begin{bmatrix} -\frac{1}{\sqrt{21}} \\ \frac{4}{\sqrt{21}} \\ \frac{2}{\sqrt{21}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} \frac{\cancel{14}7}{\cancel{6}3} \\ \frac{7}{6} \\ -7/6 \end{bmatrix} - \begin{bmatrix} -\frac{4}{21} \\ \frac{16}{21} \\ \frac{8}{21} \end{bmatrix}$$

$$= \begin{bmatrix} -1/7 \\ 1/14 \\ -3/\ldots \end{bmatrix}$$

$$\|\vec{v}_3^{\perp}\| = \sqrt{\frac{4}{196} + \frac{1}{196} + \frac{9}{196}} = \sqrt{\frac{14}{196}} = \frac{\sqrt{14}}{14}$$

$$\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \frac{14}{\sqrt{14}} \begin{bmatrix} -1/7 \\ 1/14 \\ -3/14 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{14} \\ \frac{1}{\sqrt{14}} \\ -3/\sqrt{14} \end{bmatrix}$$

$$2 + \frac{-49+4}{21} = 2 + \frac{-45}{21} = \frac{-\cancel{3}}{\cancel{21}7}$$

$$2 - \frac{147+96}{126} = 2 - \frac{243}{126} = \frac{9}{126} = \frac{3}{42} = \frac{1}{14} \qquad \frac{252}{126}$$

$$-1 - \left(\frac{-147+48}{}\right) = -1 - \frac{-99}{126} = \frac{126}{126} + \frac{99}{126} = -\frac{27}{126}$$

$$\frac{-27}{126} = \frac{-9}{42} = \frac{-3}{14}$$

$$\frac{42}{21}$$

Problem 4 (6 points). Find a basis $\mathfrak{B}$ in $\mathbb{R}^2$ such that the $\mathfrak{B}$-matrix of the vertical shear

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

is diagonal, or explain why this is not possible.

$\vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ $\quad A\vec{v}_1 = \begin{bmatrix} a \\ 4a+b \end{bmatrix}$ $\quad A\vec{v}_2 = \begin{bmatrix} c \\ 4c+d \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$

$$B = \begin{array}{cc} T(\vec{v}_1) & T(\vec{v}_2) \end{array}$$
$$B = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{array}{c} \vec{v}_1 \\ \vec{v}_2 \end{array}$$

$c_1\vec{v}_1 = A\vec{v}_1$

$\begin{bmatrix} a \\ 4a+b \end{bmatrix} = \begin{bmatrix} c_1 a \\ c_1 b \end{bmatrix}$

$\begin{bmatrix} a - c_1 a \\ 4a+b-c_1 b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} (1-c_1) & 0 & 0 \\ 4 & (1-c_1) & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & (3-c_1) & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

$\begin{bmatrix} a \\ b \end{bmatrix} = \vec{0}$

$c_2 \vec{v}_2 = A\vec{v}_2$

$\begin{bmatrix} c - c_2 c \\ 4c + d - c_2 d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

same

$\begin{bmatrix} c \\ d \end{bmatrix} = \vec{0}$

such a basis is not possible because $\vec{v}_1, \vec{v}_2 = \vec{0}$ and so are not linearly independent and cannot form a basis.
It also makes sense geometrically since a vertical shear matrix cannot be diagonal since it has the form $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, $k \in \mathbb{R}$, where it shifts the y-component, making it impossible to be a scalar multiple of $\vec{v}_1$. (ie $c_1\vec{v}_1$ and $\vec{v}_1$ are linearly independent)