Problem 1 (6 points). Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 2x_3 + x_4 \\ x_2 + 4x_3 \\ \frac{1}{2}x_1 - \frac{1}{2}x_3 + 5x_4 \\ x_1 + 3x_2 \end{bmatrix}$$

a.) (3 points) Find the matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^4$.

b.) (3 points) Let

$$\vec{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -\frac{1}{2} \end{bmatrix}$$

be a vector in $\mathbb{R}^4$. Find the solution to $T(\vec{x}) = \vec{b}$, in other words, find a vector $\vec{x}$ in $\mathbb{R}^4$ such that $A\vec{x} = \vec{b}$.

(a) $A = \begin{bmatrix} 2 & 0 & -2 & 1 \\ 0 & 1 & 4 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & 5 \\ 1 & 3 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{535}{418} \\ \frac{276}{209} \\ \frac{2981}{4598} \\ \frac{-5}{19} \end{bmatrix}$

(b)
$$\left[\begin{array}{cccc|c} 2 & 0 & -2 & 1 & 1 \\ 0 & 1 & 4 & 0 & 2 \\ \frac{1}{2} & 0 & -\frac{1}{2} & 5 & -1 \\ 1 & 3 & 0 & 0 & -\frac{1}{2} \end{array}\right]$$

$\times 2$
$$\left[\begin{array}{cccc|c} 2 & 0 & -2 & 1 & 1 \\ 0 & 1 & 4 & 0 & 2 \\ 1 & 0 & -1 & 10 & -2 \\ 1 & 3 & 0 & 0 & -\frac{1}{2} \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 10 & -2 \\ 0 & 1 & 4 & 0 & 2 \\ 2 & 0 & -2 & 1 & 1 \\ 1 & 3 & 0 & 0 & -\frac{1}{2} \end{array}\right]$$

$$\begin{array}{c} \\ \\ -2\mathrm{I} \\ -\mathrm{I} \end{array}\left[\begin{array}{cccc|c} 1 & 0 & -1 & 10 & -2 \\ 0 & 1 & 4 & 0 & 2 \\ 0 & 0 & 0 & -19 & 5 \\ 0 & 3 & 1 & -10 & \frac{3}{2} \end{array}\right]$$

$$\begin{array}{c} \\ \\ \\ -3\mathrm{II} \end{array}\left[\begin{array}{cccc|c} 1 & 0 & -1 & 10 & -2 \\ 0 & 1 & 4 & 0 & 2 \\ 0 & 0 & 0 & -19 & 5 \\ 0 & 0 & -11 & -10 & -\frac{9}{2} \end{array}\right]$$

$$\begin{array}{c} +\mathrm{III} \\ -4\mathrm{III} \\ \div -11 \\ \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{120}{11} & -\frac{35}{22} \\ 0 & 1 & 0 & -\frac{40}{11} & \frac{4}{11} \\ 0 & 0 & 1 & \frac{10}{11} & \frac{9}{22} \\ 0 & 0 & 0 & -19 & 5 \end{array}\right]$$

$$\begin{array}{c} -\frac{120}{11}\mathrm{IV} \\ +\frac{40}{11}\mathrm{IV} \\ -\frac{10}{11}\mathrm{IV} \\ \div -19 \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{535}{418} \\ 0 & 1 & 0 & 0 & \frac{276}{209} \\ 0 & 0 & 1 & 0 & \frac{2981}{4598} \\ 0 & 0 & 0 & 1 & \frac{-5}{19} \end{array}\right]$$

$\frac{4}{11} - \frac{40}{11} \cdot \frac{-5}{19}$

$= \frac{4}{11}\left(1 - 10 \cdot \frac{-5}{19}\right)$

$= \frac{4}{11}\left(1 + \frac{50}{19}\right)$

$= \frac{4}{11}\left(\frac{69}{19}\right)$

$= \frac{276}{209}$

$-\frac{35}{22} + \frac{120}{11} \cdot \frac{5}{19}$

$= \frac{1}{11}\left(\frac{-35}{2} + \frac{120 \cdot 5}{19}\right)$

$= \frac{1}{11}\left(\frac{-665 + 1200}{38}\right) = \frac{535}{418}$

$\frac{9}{22} + \frac{50}{209}$

$\frac{9}{22} + \frac{10}{11} \cdot \frac{5}{19}$

$\frac{1881 + 1100}{4598}$

$$
\begin{array}{c}
\\ \\ \times 4 \\ \\
\end{array}
\left[\begin{array}{cccc|c}
2 & 0 & -2 & 1 & 1 \\
0 & 1 & 4 & 0 & 2 \\
\frac{1}{2} & 0 & \frac{-1}{2} & 5 & -1 \\
1 & 3 & 0 & 0 & -\frac{1}{2}
\end{array}\right]
$$

$$
\left[\begin{array}{cccc|c}
2 & 0 & -2 & 1 & 1 \\
0 & 1 & 4 & 0 & 2 \\
2 & 0 & -2 & 20 & -4 \\
1 & 3 & 0 & 0 & -\frac{1}{2}
\end{array}\right]
$$

$$
\left[\begin{array}{cccc|c}
1 & 3 & 0 & 0 & -\frac{1}{2} \\
0 & 1 & 4 & 0 & 2 \\
2 & 0 & -2 & 1 & 1 \\
2 & 0 & -2 & 20 & -4
\end{array}\right]
$$

$$
\begin{array}{c}
\\ \\ \\ -\mathrm{I}
\end{array}
\left[\begin{array}{cccc|c}
1 & 3 & 0 & 0 & -\frac{1}{2} \\
0 & 1 & 4 & 0 & 2 \\
0 & 0 & 0 & -19 & 5 \\
0 & -6 & -2 & 20 & -\frac{7}{2}
\end{array}\right]
$$

$$
\begin{array}{c}
\\ -3\mathrm{II} \\ +6\mathrm{II} \\ \\
\end{array}
\left[\begin{array}{cccc|c}
1 & 0 & -12 & 0 & \frac{-13}{2} \\
0 & 1 & 4 & 0 & 2 \\
0 & 0 & 10 & 20 & \frac{17}{2} \\
0 & 0 & 0 & -19 & 5
\end{array}\right]
$$

$$
\begin{array}{c}
+12\mathrm{III} \\ -4\mathrm{III} \\ \\ \\
\end{array}
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 24 & \frac{-7}{5} \\
0 & 1 & 0 & -8 & -\frac{7}{5} \\
0 & 0 & 1 & 2 & \frac{17}{20} \\
0 & 0 & 0 & -19 & 5
\end{array}\right]
$$

$$
\begin{array}{c}
-24\mathrm{IV} \\ +8\mathrm{IV} \\ -2\mathrm{IV} \\ \\
\end{array}
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & \\
0 & 1 & 0 & 0 & \frac{-333}{95} \\
0 & 0 & 1 & 0 & \frac{523}{380} \\
0 & 0 & 0 & 1 & \frac{-5}{19}
\end{array}\right]
$$

$\frac{-7}{5} + 24 \cdot \frac{5}{19} = \frac{-3}{95}$

$\frac{-7}{5} - \frac{40}{19}$

$\frac{-133-200}{95} = \frac{-333}{95}$

$-\frac{13}{2} + \frac{3}{\cancel{12}} \cdot \frac{17}{\cancel{20}_5}$

$\frac{-65+51}{10}$

3

Problem 2 (6 points). Are the following matrices in reduced row echelon form? If not, find the reduced row echelon form.

a.) (3 points) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 4 & \frac{1}{2} \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 4 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{1}{8} \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} -2\text{III} \\ -\frac{1}{8}\text{III} \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b.) (3 points) $B = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$-8\text{III} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rref}(B) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 3 (6 points). For each of the matrices below, either find an inverse or explain why no inverse exists.

a.) (3 points) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -2\text{I} \\ -3\text{I} \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ -2\text{II} \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$-3-2(-2)$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

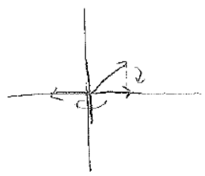
b.) (3 points) $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 7 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -\text{I} \\ -\text{I} \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 6 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} -\text{II} \\ \\ -3\text{II} \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 & -3 & 1 \end{array}\right]$$

No inverse exists because the rref(B) has a row of zeros, meaning the transformation associated either has no solutions or infinite solutions. In either case, there can't be one unique solution, so the inverse of the matrix B cannot exist.

Problem 4 (6 points). Find a matrix which describes the projection on the horizontal line combined with a reflection about the vertical line in $\mathbb{R}^2$. Does the order of these transformations matter?

Let A be matrix that projects a vector $\in \mathbb{R}^2$ on the horizontal line and B a matrix that reflects about the vertical line.



$$A = \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix} \qquad \text{Let } \vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ since on horizontal line.}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix} \qquad T(\vec{e}_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad T(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \qquad BA = AB \text{ in this case.} \qquad = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

The order of the transformations does not matter. The matrix that describes the combination is $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$