1. Provide answers to the following questions. **No justification is required.**

(a) (2 points) How many solutions does the system $\begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ -12 \\ 3 \\ 0 \end{bmatrix}$ have?

Infinity.

Since $\begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & 0 & -12 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a rref and x_2, x_5 are free variables.

(b) (2 points) Is there a 4×2 matrix with rank 4?

No.

Consider the rref of this matrix. There are at most 2 leading ones.

(c) (2 points) Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ 2x + 1 \end{bmatrix}$. Is T linear?

No.

Linear transformation maps $\vec{0}$ to $\vec{0}$.

But $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(d) (2 points) Describe the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\vec{x}) = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \vec{x}$ geometrically.

T is of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, which must be a combination of a scaling and a rotation.

Here $a=-1, b=1$, $a^2+b^2=2$. The scaler $=\sqrt{2}$,

and the rotation matrix $= \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$, which means to rotate 135° counterclockwisely.

2. (a) (4 points) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 2 \\ 4 & 2 & 9 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 4 & 2 & 9 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \mathrm{II} - \mathrm{I} \\ \mathrm{III} - 4\times\mathrm{I} \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -3 & -4 & 0 & 1 \end{array}\right]$$

$$\mathrm{III} - 2\times\mathrm{II} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -2 & -2 & 1 \end{array}\right]$$

$$\begin{array}{l} \mathrm{III} \times (-1) \\ \mathrm{II} + \mathrm{III} \\ \mathrm{I} - 3\times\mathrm{III} \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -6 & 3 \\ 0 & 1 & 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 2 & 2 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -5 & -6 & 3 \\ 1 & 3 & -1 \\ 2 & 2 & -1 \end{bmatrix}.$$

(b) (3 points) Find a vector $\vec{x} \in \mathbb{R}^3$ such that $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Is it unique?

Since A^{-1} exists, there exists a unique solution $\vec{x} = A^{-1}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 & -6 & 3 \\ 1 & 3 & -1 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

3. (a) (3 points) Find the matrix of the reflection $R : \mathbb{R}^2 \to \mathbb{R}^2$ that transforms $\begin{bmatrix}1\\0\end{bmatrix}$ into $\begin{bmatrix}0\\-1\end{bmatrix}$.

Let A be the matrix of the reflection R.

$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. $a^2+b^2=1$.

$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

We have $A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$,

then $a=0$, $b=-1$.

$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

(b) (2 points) Find the matrix of the composition $R^2 = R \circ R$.

The matrix of R^2 is A^2:

Method 1: $A^2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Method 2: Doing a reflection twice is the identity.

R is a reflection in $\mathbb{R}^2$.

Hence $R^2 = I_2$.

(c) (6 points) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation that satisfies

$$T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Find the matrix representing T.

B is the matrix representing T:

$B\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}.$

$\begin{bmatrix} 4 & 5 \\ -1 & 0 \end{bmatrix}.$

As $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{\det\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$

Then $B = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -1 & 0 \end{bmatrix}.$

(d) (4 points) Find the matrix of the composition $R^{-1} \circ T$.

Since $R^2 = I_2$, we have $R^{-1} = R$.

Then $R^{-1} \circ T = R \circ T$

$\begin{bmatrix} 1 & 0 \\ -4 & -5 \end{bmatrix}.$

The matrix of $R \circ T$: $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & -5 \end{bmatrix}.$

4. (a) (4 points) For which values of a, b the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 1 \\ 0 & 0 & b \end{bmatrix}$ is invertible?

If $a=0$, then the second column of A is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Hence there can't be a leading one in this column of rref(A).
Then rank(A) ≤ 2. A is not invertible.

If $b=0$, then the third row of A is $[0,0,0]$, Hence there can't be a leading one in this row of rref(A).
Then rank(A) ≤ 2. A is not invertible.

When $a\neq 0$, $b \neq 0$,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & a & 1 & 0 & 1 & 0 \\ 0 & 0 & b & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{\text{II} \times \frac{1}{a} \\ \text{III} \times \frac{1}{b}}]{\substack{\text{I} - \frac{1}{b}\times \text{III} \\ \text{II} - \frac{1}{b}\times\text{III}}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -\frac{1}{b} \\ 0 & 1 & 0 & 0 & \frac{1}{a} & -\frac{1}{ab} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{b} \end{array}\right]$$, then $A^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{b} \\ 0 & \frac{1}{a} & -\frac{1}{ab} \\ 0 & 0 & \frac{1}{b} \end{bmatrix}$.

A is invertible.

(b) (6 points) Consider the system $A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. For which values of a, b is there a unique solution? infinitely many? none?

If A is invertible, since A^{-1} exists,
$\vec{x} = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is the unique solution.
Hence $a\neq 0$, $b \neq 0 \Rightarrow$ unique solution.

If $a=0$, $b\neq 0$,

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & b & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$, the third row $\Rightarrow$ no solution.

If $b=0$, $a\neq 0$,

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & a & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1/a & 1/a \\ 0 & 0 & 0 & 0 \end{array}\right]$$, infinity many solutions
(x_3 is a free variable).

If $a=b=0$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right],$$ infinity many solutions (x_2 is a free variable)

In summary,

$a \neq 0, b \neq 0 \Rightarrow$ unique solution

$a = 0, b \neq 0 \Rightarrow$ no solution

$b = 0 \Rightarrow$ infinity many solutions.