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MATH 33A-3: Linear Algebra Final Exam

This exam contains 18 pages (including this cover page) and 13 problems. The following rules apply:

- **You have 24 hours to submit this take home exam. The deadline for submission is Thursday, March 19, 6:59am (PDT).** Please, note that this deadline applies to all students, including those registered with CAE. **No late submission of the exam will be possible.**
- **Collaborations on the final are not allowed.** You are under strict instructions not to discuss the exam or questions related to the exam with anybody. Please, be reminded of the Student Conduct Code (it can be found at www.deanofstudents.ucla.edu; see, in particular, Section 102.01 on academic dishonesty).
- You are **allowed to use the textbook and your notes** from the class, as well as, the notes posted by the instructor and/or TA's, while working on the exam. **You should not use any other resources, including online ones.**
- **If you use a result from class, discussion session, the textbook, lecture notes, or a homework/midterm, you must indicate this,** reference the source, and explain why the result may be applied.
- **You should not use computing systems (e.g., Mathematica or Matlab) while working on the problems.**
- Show your work on each problem. **All answers must be justified, unless otherwise mentioned. Mysterious or unsupported answers will not receive credit.** A correct answer, unsupported by calculations and/or explanation will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **Organize your work,** in a reasonably neat and coherent way. Work scattered all over the page without a clear ordering will receive very little credit.
- **I reserve the right to contact the students after the exam and ask for additional explanations of solutions for problems on the final.**
- **Deviations from the announced rules may render the exam void.**
- You can choose one of the following two options for submission of your exam. In both options, you should use regular size (preferably) white paper. Please, follow the instructions below precisely, as failing to do so may result in loosing parts of your solutions!

Option I (if you have access to printer):

- You print the pdf file with the exam (all pages, 1-sided).
- Fill in the information (Name and UID) and sign at the top of the cover page agreeing to the rules.
- Write all your solutions on the printed exam (using only the front side of the pages).
- Scan all the pages of the exam. The pdf file with your solutions should have the same number of pages as the original pdf file (i.e., your file should have 18 pages).
- Submit the scanned pdf file to Gradescope via the course CCLE webpage.

Option II (if you do not have access to printer):

- Open the file with the exam on your electronic device.
- The first page of your solution should include your name (printed) followed by your UCLA ID (also printed) on the top of the page. Below, please include the following statement, followed by your signature and date (the first page should not contain anything else, i.e., no solutions on the first page):
  "I assert, on my honor, that I have not received assistance of any kind from any other person, and have not used any non-permitted materials or technologies while working on the final. I agree with the rules summarized on the exam assignment cover page"
- The remaining part of your solution should be formatted in the same way as the original pdf file with the exam. This means, if the original file have Problem X on page Y, then you should also have Problem X on page Y in your solution (even if you did not do it).
- Scan all the pages of the exam. The pdf file with your solutions should have the same number of pages as the original pdf file (i.e., your file should have 18 pages).
- Submit the scanned pdf file to Gradescope via the course CCLE webpage.

Good luck!

1. Provide answers to the following questions. **No justification is required.**

(a) (2 points) Suppose that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are three linear independent vectors in $\mathbb{R}^5$. Let $\vec{w}_1 = 2\vec{v}_1$, $\vec{w}_2 = \vec{v}_1 + \vec{v}_2$, and $\vec{w}_3 = \vec{v}_2 - \vec{v}_3$. What is rref $\begin{bmatrix} | & | & | \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \\ | & | & | \end{bmatrix}$? $c_1(2\vec{v}_1)+c_2(\vec{v}_1+\vec{v}_2)+c_3(\vec{v}_2-\vec{v}_3)=0$

$(2c_1+c_2)\vec{v}_1 + (c_2+c_3)\vec{v}_2 - c_3\vec{v}_3 = 0$

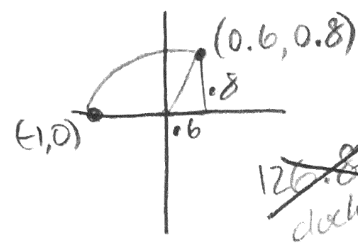
$-c_3=0,\ c_2+c_3=0,\ c_2=0,\ 2c_1+c_2=0,\ c_1=0$

Thus, $\vec{w}_1, \vec{w}_2$, and $\vec{w}_3$ are linearly independent.

$$\text{rref}\begin{bmatrix} | & | & | \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) (2 points) Find the matrix of the rotation in the plane that transforms $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ into $\begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$.

rotation: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix}$



~~126.87° clockwise~~

$$\begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix}$$

(c) (2 points) Let A be a 3×5 matrix. What are the possible values of $\dim(\text{Ker}(A))$?

$\text{rank}(A) = 0,1,2,3$ $\quad \dim(\text{Ker}(A)) = 5 - \text{rank}(A) = 5,4,3,2$

The possible values of $\dim(\text{Ker}(A))$ are $2,3,4,5$

(d) (2 points) Are the matrices $\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$ and $\begin{bmatrix} 2 & -2 \\ 1 & -3 \end{bmatrix}$ similar?

Looking for $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $AS = SB$

$A = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & -2 \\ 1 & -3 \end{bmatrix}$ $\quad \det(A) \neq \det(B)$

They are <u>not</u> similar

$\det(A) = -2-3 = -5$

$\det(B) = -6-2 = -4$

2. Let A be a square matrix. For each of the following statements determine whether it is true or false. **No justification is required.**

   (a) (1 point) If A has orthonormal rows, then A is orthogonal.

   True

   (b) (1 point) If A is diagonalizable, then A^2 is diagonalizable as well.

   True

   (c) (1 point) If A^2 is diagonalizable, then A is diagonalizable as well.

   False

   If $A^2 = 0$ and A is not 0-matrix

   (d) (1 point) $A + A^T$ is always diagonalizable.

   $A + A^T$ is symmetric

   True

   (e) (1 point) $A - A^T$ is always diagonalizable.

   $A - A^T$ is skew-symmetric

   $\boxed{\text{False}}$

   (f) (1 point) If A is symmetric, then the eigenvalues of A coincide with the singular values of A.

   False

   Not if λ are negative, as singular value $= |\lambda|$

3. (3 points) Let $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ and B be an orthogonal 2×2 matrix. Find $\det(A^{-2}B^3A^TAB^TA^2)$.

Show work/Justify your answer to get full credit.

Theorem 6.2.6 from textbook: If A and B are nxn matrices and m is a positive integer,

① $\det(AB) = (\det A)(\det B)$

② $\det(A^m) = (\det A)^m$

Theorem 6.2.8 from textbook: If A is invertible, then

③ $\det(A^{-1}) = \frac{1}{\det A} = (\det A)^{-1}$

$\det(A^{-2}B^3A^TAB^TA^2) = \det(A^{-2})\det(B^3)\det(A^T)\det(A)\det(B^T)\det(A^2)$ by ①

= Determinants are scalars, so by the commutative property and ①

$= \det(A^{-2})\det(B^3B^T)\det(A^T)\det(A)\det(A^2)$

$B^T = B^{-1}$ for an orthogonal nxn matrix B, so

$= \det(A^{-2})\det(B^3B^{-1})\det(A^T)\det(A)\det(A^2)$

$= \det(A^{-2})\det(B^2)\det(A^T)\det(A)\det(A^2)$

Theorem 6.3.1: The determinant of an orthogonal matrix is 1 or −1, so

① → $\det(B^2) = (\det B)^2 = 1^2$ or $(-1)^2 = 1$

$\det(A^{-2})\det(A^T)\det(A)\det(A^2)$

A is a square matrix, so $\det(A^T) = \det A$

$= \det(A^{-2})\underbrace{\det(A)\det(A)}_{①}\det(A^2) = \det(A^{-2})\det(A^2)\det(A^2)$

by ①

$= \det(\cancel{A^{-2}} \cdot \cancel{A^2} \cdot A^2) = \det(A)^2$

$\det(A) = 8-3 = 5$

$(\det(A))^2 = 5^2 = \boxed{25}$

4. (4 points) Let A be the matrix of the reflection about the plane $x_1 - 2x_2 + 3x_3 = 0$ in $\mathbb{R}^3$. Let $B = 2A - I_3$ (here, as ususal, I_3 denotes the identity matrix of size 3×3). Is B diagonalizable? If yes, find a diagonal matrix D that is similar to B. **Justify your answer!**

$\text{ref}_V(\vec{x}) = \vec{x} - 2(\vec{u}\cdot\vec{x})\vec{u}$ $\vec{n}$ = Normal vector $= \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ $\vec{u} = \frac{\vec{n}}{\|\vec{n}\|} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

$$\text{ref}_V(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{\sqrt{14}\sqrt{14}}\left(\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix}$$

$$\text{ref}_V(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{7}\left(\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{2}{7}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$$

$$\text{ref}_V(\vec{e}_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{7}\left(\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{7}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -3 \\ 6 \\ -2 \end{bmatrix}$$

$$A = \frac{1}{7}\begin{bmatrix} 6 & 2 & -3 \\ 2 & 3 & 6 \\ -3 & 6 & -2 \end{bmatrix} \qquad B = 2A - I_3 = \frac{1}{7}\left(\begin{bmatrix} 12 & 4 & -6 \\ 4 & 6 & 12 \\ -6 & 12 & -4 \end{bmatrix} - \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}\right)$$

$$B = \frac{1}{7}\begin{bmatrix} 12 & 4 & -6 \\ 4 & 6 & 12 \\ -6 & 12 & -4 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 5 & 4 & -6 \\ 4 & -1 & 12 \\ -6 & 12 & -11 \end{bmatrix}$$

$\boxed{\text{According to the Spectral theorem, since } B \text{ is symmetric } (B^T = B), B \text{ is orthogonally diagonalizable, and thus \underline{IS Diagonalizable}}}$

$$f_B(\lambda) = \det(B - \lambda I_n) = 0 = \det\begin{bmatrix} \frac{5}{7}-\lambda & \frac{4}{7} & -\frac{6}{7} \\ 4/7 & -\frac{1}{7}-\lambda & 12/7 \\ -6/7 & 12/7 & -\frac{11}{7}-\lambda \end{bmatrix}$$

$$= \left(\frac{5}{7}-\lambda\right)\left[\left(-\frac{1}{7}-\lambda\right)\left(-\frac{11}{7}-\lambda\right) - \frac{144}{49}\right] - \frac{4}{7}\left[\left(\frac{4}{7}\right)\left(-\frac{11}{7}-\lambda\right) + \frac{72}{49}\right] - \frac{6}{7}\left[\frac{48}{49} - \left(\frac{6}{7}\right)\left(\lambda + \frac{1}{7}\right)\right]$$

$$= \left(\frac{5}{7}-\lambda\right)\left[\lambda^2 + \frac{12}{7}\lambda - \frac{19}{7}\right] - \frac{4}{7}\left[-\frac{4\lambda}{7} + \frac{4}{7}\right] - \frac{6}{7}\left[-\frac{6}{7}\lambda + \frac{6}{7}\right]$$

$\lambda_1 = -3,\ \lambda_2 = 1,\ \lambda_3 = 1$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$$, a diagonal matrix that is similar to B

5. Let $A = \begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & a & 1 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix}$ for some real constant a. Answer the following questions. **Show work/Justify your answers to get full credit.**

(a) (3 points) Find rref(A).

(b) (3 points) Find det(A).

(c) (2 points) Find all vectors $\vec{y}$ in $\mathbb{R}^4$ that can be written in the form $\vec{y} = A\vec{x}$ for some x in $\mathbb{R}^4$? image of A

a) $\begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & a & 1 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & a & 1 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & a & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & a & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 1-a & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1+a \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$

rref$(A) = I_4$

b) $\begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & a & 1 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & a & 1 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & a & 1 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix}$

$\det A = \sum (\text{sgn } P)(\text{prod } P)$

$= (-1)^2(3\cdot 2\cdot 1\cdot 2) + (-1)^3(3\cdot 2\cdot 1\cdot 2)$

$+ (-1)^4(3\cdot 2\cdot 1\cdot 1) = 12 - 12 + 6 = 6$

$\det(A) = 6$

c) is on next page

...continue your solution of Problem 5 here if needed...

c) in part a), rref(A) = I_4, so there are no redundant column vectors, meaning that im(A) is the column space, or

$$\text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ a \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \\ 2 \end{pmatrix} \right\}$$

All vectors $\vec{y}$ in $\mathbb{R}^4$ that can be written in the form $\vec{y} = A\vec{x}$ for some $\vec{x}$ in $\mathbb{R}^4$, is simply the image of A, so $\vec{y}$ is represented by

$$\text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ a \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \\ 2 \end{pmatrix} \right\}$$

6. Let $A = \begin{bmatrix} 1 & 1 & 2 & -4 \\ -1 & -1 & 1 & 1 \\ 2 & 2 & 2 & -6 \end{bmatrix}$. Answer the following questions. **Show work/Justify your answers to get full credit.**

(a) (3 points) Find a basis of Ker(A).

(b) (3 points) Find an orthonormal basis of Ker(A).

(c) (4 points) Find a vector $\vec{y}$ in Ker(A) minimizing the length of $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \vec{y}$. Is such a vector unique?

a) $\left(\begin{array}{cccc|c} 1 & 1 & 2 & -4 & 0 \\ -1 & -1 & 1 & 1 & 0 \\ 2 & 2 & 2 & -6 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 2 & -4 & 0 \\ 0 & 0 & 3 & -3 & 0 \\ 2 & 2 & 2 & -6 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 2 & -4 & 0 \\ 0 & 0 & 3 & -3 & 0 \\ 0 & 0 & -2 & 2 & 0 \end{array}\right)$

$\to \left(\begin{array}{cccc|c} 1 & 1 & 2 & -4 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 2 & -4 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$x_1 = -s + 2t$
$x_2 = s$
$x_3 = t$
$x_4 = t$

$s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

Basis of Ker(A) $= \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

b) Basis of Ker(A) $= \{\vec{v}_1, \vec{v}_2\}$ $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$\vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \dfrac{\vec{v}_1}{\sqrt{1+1}} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$

$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$

$\vec{u}_1 \cdot \vec{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = -\dfrac{2}{\sqrt{2}} = -\sqrt{2}$

. . . continue your solution of Problem 6 here if needed . . .

6b) continued

$$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \vec{v}_2 + \sqrt{2}\,\vec{u}_1$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \sqrt{2}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2^{\perp}}{\sqrt{4}} = \frac{\vec{v}_2^{\perp}}{2} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

Orthonormal basis of Ker(A) $= \left\{ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \right\}$

6c)

Ker A $= \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ $\quad B = \begin{bmatrix} -1 & 2 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad B^T = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{matrix} x_1 = 0 \\ x_2 = 0 \end{matrix} \; \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{x}$$

we made the columns of B the basis of Ker A, so $B\vec{x}$ = some vector in Kernel A

Thus, $\underline{\text{Ker}(B) = \{\vec{0}\}}$, so $x^* = (B^TB)^{-1}B^T\vec{b}$ (x^* minimizes $|b - A\vec{x}|$

$$x^* = \left(\begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 6 \end{bmatrix}^{-1}\begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 6 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -2 & 6 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix}$$

$$\vec{y} = Bx^* = \begin{bmatrix} -1 & 2 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \vec{y}$$

, $\vec{y}$ is a unique vector, as Ker(B) = $\{\vec{0}\}$ implies a unique least-squares solution. Each vector in Ker(A) is unique, the minimum distance from a plane is unique

7. (4 points) Use the method of least squares to find constants a and b so that the function $f(x) = a + b\sin(\pi x/2)$ gives an optimal fit for the data points $(0, 3)$, $(1, 2)$, $(5/3, 0)$, and $(3, -2)$. **Show work to get full credit.**

$f(x) = a + b\sin\left(\frac{\pi x}{2}\right)$ $\quad f(0) = 3$, $f(1) = 2$, $f\left(\frac{5}{3}\right) = 0$, $f(3) = -2$

$$\begin{aligned} a + 0 &= 3 \\ a + b &= 2 \\ a + \tfrac{1}{2}b &= 0 \\ a - b &= -2 \end{aligned}$$

$a = 3$

$a + b = 2$ $\quad a - b = -2$

$3 + b = 2$ $\quad 3 - (-1) = -2$

$b = -1$ $\quad 4 \neq -2$,

so the system is <u>inconsistent</u>

$A\begin{bmatrix} a \\ b \end{bmatrix} = \vec{b}$ where $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & \frac{1}{2} \\ 1 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -2 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & -1 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & \frac{1}{2} & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$, so $\ker(A) = \{\vec{0}\}$, Therefore...

$$\begin{bmatrix} a^* \\ b^* \end{bmatrix} = (A^TA)^{-1}A^T\vec{b} = \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & \frac{1}{2} \\ 1 & -1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 0 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & \frac{1}{2} \\ \frac{1}{2} & \frac{9}{4} \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 0 \\ -2 \end{bmatrix} = \frac{4}{35}\begin{bmatrix} \frac{9}{4} & -\frac{1}{2} \\ -\frac{1}{2} & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 0 \\ -2 \end{bmatrix}$$

$$= \frac{4}{35}\begin{bmatrix} \frac{9}{4} & \frac{7}{4} & 2 & \frac{11}{4} \\ -\frac{1}{2} & \frac{7}{2} & \frac{3}{2} & -\frac{9}{2} \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 0 \\ -2 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 9 & 7 & 8 & 11 \\ -2 & 14 & 6 & -18 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 0 \\ -2 \end{bmatrix}$$

$$= \frac{1}{35}\begin{bmatrix} 19 \\ 58 \end{bmatrix}$$

$$\boxed{a = \frac{19}{35},\ b = \frac{58}{35}}$$

8. Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$. Answer the following questions. **Show work/Justify your answers to get full credit.**

(a) (3 points) Find eigenvalues of A and determine their algebraic and geometric multiplicities.

(b) (3 points) Is A diagonalizable? If so, find an invertible matrix S and a diagonal matrix B such that $A = SBS^{-1}$.

(c) (2 points) Does A have an orthonormal eigenbasis? Yes or No, justification required

(d) (4 points) Find $A^5 \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}$.

a) $f_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 0-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & -1-\lambda \end{bmatrix} = -\lambda(1-\lambda)(-1-\lambda) + 1(0 - 2(1-\lambda))$

$= \lambda(1+\lambda)(1-\lambda) - 2(1-\lambda) = (1-\lambda)(\lambda^2+\lambda-2) = 0$

$-(\lambda-1)^2(\lambda+2) = 0 \quad \lambda = -2, 1 \leftarrow$ multiplicity 2

$\lambda = -2$: $\left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad x_1 = -\frac{1}{2}s$, $x_2 = 0$, $x_3 = s$ $\quad s\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$

gemu$(-2) = \dim(E_{-2}) = 1$

$\lambda = 1$: $\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = t$, $x_2 = s$, $x_3 = t$ $\quad s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

gemu$(1) = \dim(E_1) = 2$

$\lambda_1 = -2$, almu$(-2) = 1$, gemu$(-2) = 1$

$\lambda_2 = 1$, almu$(1) = 2$, gemu$(1) = 2$

b, c, d on next page

... continue your solution of Problem 8 here if needed ...

8.

b) A is a 3×3 matrix, so it is diagonalizable if and only if the geometric multiplicities add up to n, or 3 (ie. if s=n), according to Theorem 7.3.3 of the textbook.

gemu(−2) + gemu(1) = 1+2=3, so A is diagonalizable.

$A = SBS^{-1}$... $S = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}$, $B = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

where $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are the eigenvectors from a) and λ_i is the associated eigenvalue with $\vec{v}_i$

$S = \begin{bmatrix} -\frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) No. It can only have an orthonormal eigenbasis if and only if it is symmetric ($A^T = A$), according to theorem 8.1.1 of the textbook.

$A^T = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \neq A$, so $A^T \neq A$.

d) $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$ $A^3 = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 3 \\ 0 & 1 & 0 \\ 6 & 0 & -5 \end{bmatrix}$

$A^4 = \begin{bmatrix} -2 & 0 & 3 \\ 0 & 1 & 0 \\ 6 & 0 & -5 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & -5 \\ 0 & 1 & 0 \\ -10 & 0 & 11 \end{bmatrix}$ $A^5 = \begin{bmatrix} 6 & 0 & -5 \\ 0 & 1 & 0 \\ -10 & 0 & 11 \end{bmatrix}$

$A^5 = \begin{bmatrix} 6 & 0 & -5 \\ 0 & 1 & 0 \\ -10 & 0 & 11 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -10 & 0 & 11 \\ 0 & 1 & 0 \\ 22 & 0 & -21 \end{bmatrix}$

$A^5 \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} -10 & 0 & 11 \\ 0 & 1 & 0 \\ 22 & 0 & -21 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} -33 \\ 3 \\ 63 \end{bmatrix}$

9. (4 points) Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$, that is, A is the matrix from Problem 8. Is A similar to $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Justify your answer!

First, we prove that two similar matrices have the same characteristic polynomial. If C and D are similar, we can denote $D = S^{-1}CS$ where S is invertible

$CS = SD$ $D = S^{-1}CS$, if C and D are similar $n \times n$ matrices

$$f_D(\lambda) = \det(D - \lambda I) = \det(S^{-1}CS - \lambda I) = \det(S^{-1}CS - \lambda S^{-1}S)$$
$$= \det(S^{-1}(C - \lambda I)S) = \det(S^{-1})\det(C - \lambda I)\det(S)$$

Determinants are scalars, so the commutative property applies...

Also, $\det(S^{-1}) = \frac{1}{\det S} = (\det S)^{-1}$

$$\det(S^{-1})\det(C - \lambda I)\det(S) = \cancel{\frac{1}{\det S}}\,\cancel{\det S}\,\det(C - \lambda I)$$
$$= \det(C - \lambda I) = f_C(\lambda)$$

Thus, B and A should have the same characteristic polynomials, if they are similar

$$f_B(\lambda) = \det(B - \lambda I) = \det\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda)(-2-\lambda)$$
$$= (1 - \lambda - \lambda + \lambda^2)(-2-\lambda) = (\lambda^2 - 2\lambda + 1)(-2-\lambda) = \cancel{-2\lambda^2} + 4\lambda - 2 - \lambda^3 + \cancel{2\lambda^2} - \lambda$$
$$= -\lambda^3 + 3\lambda - 2$$

$$f_A(\lambda) = (1-\lambda)(\lambda^2 + \lambda - 2) = \cancel{\lambda^2} + \lambda - 2 - \lambda^3 - \cancel{\lambda^2} + 2\lambda = -\lambda^3 + 3\lambda - 2 \leftarrow \text{From \#8}$$

$f_A(\lambda) = f_B(\lambda)$, meaning that A and B have the same eigenvalues and same algebraic multiplicities for each eigenvalue, so

A and B are similar.

10. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$. Your goal is to compute the singular value decomposition $A = U\Sigma V^T$.

Answer the following questions. **Show work to get full credit.**

(a) (3 points) Find the singular values of A.

(b) (1 point) Find Σ.

(c) (3 points) Find V.

(d) (4 points) Find U.

a) A^TA is symmetric, as $(A^TA)^T = A^T(A^T)^T = A^TA$.

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

$$f_{A^TA}(\lambda) = \det(A^TA - \lambda I) = \det\begin{bmatrix} 2-\lambda & 2 \\ 2 & 5-\lambda \end{bmatrix} = (2-\lambda)(5-\lambda) - 4 = 0$$

$$(2-\lambda)(5-\lambda) - 4 = 10 - 7\lambda + \lambda^2 - 4 = \lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda-1)(\lambda-6) = 0 \quad \lambda = 1, 6 \quad \sigma_1 = \sqrt{\lambda_1} = \sqrt{1} = 1$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{6} = \sqrt{6}$$

$$\boxed{\sigma_1 = 1, \ \sigma_2 = \sqrt{6}}$$

b) Σ is a $n \times m$ (3×2) matrix whose first r diagonal entries are $\sigma_1, \ldots, \sigma_r$.

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{6} \\ 0 & 0 \end{bmatrix}$$

c)

$$\lambda_1 = 1: A^TA - 1I = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = -2s \\ x_2 = s \end{array} \quad \vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$\lambda_2 = 6: A^TA - 6I = \left[\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{l} x_1 = \frac{1}{2}s \\ x_2 = s \end{array} \quad \vec{v}_2 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \quad \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{2}{\sqrt{5}}\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

. . . continue your solution of Problem 10 here if needed. . .

c) continued

$$\boxed{V = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}}$$

d) Compute U by the formula $u_i = \frac{1}{\sigma_i} A \vec{v}_i$

$$\vec{u}_1 = \frac{1}{\sigma_1} A \vec{v}_1 = \frac{1}{1}\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} A \vec{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{6}}\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 5/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$U = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 & 5/\sqrt{6} \\ -1 & -2/\sqrt{6} \\ -2 & 1/\sqrt{6} \end{bmatrix}$$

11. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$, that is, A is the same matrix as in Problem 10. Also let $\vec{w}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Answer the following questions. **Show work/Justify your answers to get full credit.**

(a) (2 points) Find the area of the triangle Δ between $\vec{w}_1$ and $\vec{w}_2$.

(b) (3 points) What is the area of the image of the triangle Δ (between $\vec{w}_1$ and $\vec{w}_2$) under A?

a) $\left|\det\begin{bmatrix} \vec{w}_1 & \vec{w}_2 \end{bmatrix}\right| = \left|\det\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}\right| = |2-3| = +1 =$ Area of parallelogram $= 1$

Area of Δ between $\vec{w}_1$ and $\vec{w}_2 = \frac{1}{2}$(Area of parallelogram)

$= \frac{1}{2}(1) = \boxed{\frac{1}{2} \text{ units}^2}$

b) ~~det(A) is how much area changes by / the expansion factor~~

~~$V(A\vec{v}_1, \ldots, A\vec{v}_n) = |\det A| \, V(\vec{v}_1, \ldots, \vec{v}_n)$~~

$A\vec{w}_1 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \vec{v}_1 \quad A\vec{w}_2 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$

Area of im $\Delta = \frac{1}{2}\|\vec{v}_1 \times \vec{v}_2\|$

$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 4 & -1 & 2 \\ 5 & -1 & 3 \end{vmatrix} = (-3+2)i - (12-10)j + (-4+5)k = -1i - 2j + 1k$

$= \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

$\|\vec{v}_1 \times \vec{v}_2\| = \sqrt{1+4+1} = \sqrt{6}$

Area of im $\Delta = \frac{1}{2}\sqrt{6} = \frac{\sqrt{6}}{2}$ units^2

12. (6 points) Determine **all** real values of a, b, c for which the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ a & 3 & 0 \\ b & 1 & c \end{bmatrix}$ is diagonalizable. **Show work/Justify your answer to get full credit.**

$f_A(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} 2-\lambda & 0 & 0 \\ a & 3-\lambda & 0 \\ b & 1 & c-\lambda \end{bmatrix} = (2-\lambda)(3-\lambda)(c-\lambda) = 0$

Therefore, $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = c$. There are 3 cases to consider: $c=2, c=3, c \neq 2 \;\&\; c \neq 3$

case 1, $c=2$: $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 2$

$\lambda_{1,3} = 2$; $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 1 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} a & 1 & 0 & 0 \\ b & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

if $a \neq b$, and $a = k_1$, $b = k_2$, where k_1 and k_2 are real-valued scalars

$\left(\begin{array}{ccc|c} k_1 & 1 & 0 & 0 \\ k_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1/k_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = s \end{matrix}$ $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

There is only one eigenvector, but the algebraic multiplicity is 2, so this is NOT Diagonalizable

if $a \neq b$, $a = b = k$, where k is a real-valued scalar

$\left(\begin{array}{ccc|c} k & 1 & 0 & 0 \\ k & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} k & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1/k & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\begin{matrix} x_1 = -\frac{1}{k}s \\ x_2 = s \\ x_3 = t \end{matrix}$ $\vec{v}_1 = \begin{bmatrix} -1/k \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

There are 2 eigenvectors for $\lambda_{1,3} = 2$, so gemu$(2) = 2$. gemu$(3) = 1$ so $\sum$ gemu$(\lambda) = 3 = n$, so we can form an eigenbasis. This subcase IS Diagonalizable

case 2, $c=3$: $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 3$

$\lambda_{2,3} = 3$; $\left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & 1 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ Regardless of what values a and b are, so... $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Again, only one eigenvector associated with $\lambda_{2,3} = 3$, but algebraic multiplicity is 2, so this case is NOT Diagonalizable.

case 3, $c \neq 2, c \neq 3$: $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = k$, where k is a real-valued scalar not equal to 2 or 3.

Thus, we have 3 distinct eigenvalues for a 3x3 matrix, so by Thm 7.3.4, A is diagonalizable as we can form an eigenbasis with one eigenvector from each eigenvalue.

SOLUTION: A IS Diagonalizable when...
1) $c = 2$, $a \neq b$
2) $c \neq 2$ and $c \neq 3$

13. (6 points) Suppose A is a symmetric $n \times n$ matrix that satisfies $A^3 = I_n$, where I_n is is the identity matrix of size $n \times n$. **Prove** that $A = I_n$.

Hint: If $A^3 = I_n$, what are the possible eigenvalues of A?

A is symmetric, so it is orthogonally diagonalizable by the spectral theorem (textbook), meaning there is an orthonormal eigenbasis.

If $\vec{x}$ is an eigenvector of A, $A\vec{x} = \lambda\vec{x}$.

Applying A twice, we get $A^3\vec{x} = \lambda^3\vec{x}$

It is given that $A^3 = I_n$, so $\vec{x} = \lambda^3\vec{x}$

Multiplying by inverse of $\vec{x}$, on both sides, implying that $\lambda^3 = 1$. Therefore, all eigenvalues associated with A are 1.

Since there is an orthonormal eigenbasis for A, all vectors of A can be represented by a linear combination of eigenvectors.

Namely, for an orthonormal eigenbasis B of eigenvectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$, we know some vector $w = c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_n\vec{u}_n$

Thus, $A\vec{w} = A(c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_n\vec{u}_n) = c_1 A\vec{u}_1 + c_2 A\vec{u}_2 + \dots + c_n A\vec{u}_n$

Since we know that $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ are eigenvectors of A, and the associated eigenvalues are 1 ($\lambda = 1$), we must know that $A(c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_n\vec{u}_n) = c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_n\vec{u}_n$

This shows that any vector, $\vec{w}$, applied with A is sent to the vector itself, so A must be I_n.

In other words, $A\vec{w} = \vec{w}$, so $A = I_n$.