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2	4
3	4
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MATH 33A Midterm II, Spring 2018

[redacted] Section Number: Breen 1A 1B,
1D

Instruction: Justify all your answers. No points will be given without sufficient reasoning/calculations.

Problem 1. (4)

(i) Use row or column reduction to compute $det A$, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 1 & 4 & 5 & 7 \end{bmatrix}.$$

(ii) Find $det B$, where

$$B = \begin{bmatrix} 0 & 0 & a & 4 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & b & c \\ 1 & d & e & f \end{bmatrix}.$$

Here a, b, c, d, e and f are constants.

(iii) Find $det(AB)$, $det(A^{-1})$ and $det(B^{-1})$.

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i) $\det A = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 4 \\ 0 & 3 & 4 & 6 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & -3 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$= (1)(1)(-1)(1) = \boxed{-1}$

ii) $\det B = \begin{pmatrix} 1 & d & e & f \\ 0 & 2 & b & c \\ 0 & 0 & 3 & 0 \\ 0 & 0 & a & 4 \end{pmatrix} = (2)(12) = \boxed{24}$

iii) $\det(AB) = (\det A)(\det B) = \boxed{-24}$

$\det(A^{-1}) = \frac{1}{-1} = \boxed{-1}$

$\det(B^{-1}) = \boxed{\frac{1}{24}}$

Problem 2. (4)

Let V =span $\{\vec{v_1}, \vec{v_2}\}$ be a subspace in $\mathbf{R}^3$, where $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(i) Find an orthonormal basis for V by using Gram-Schmidt process.

(ii) Find the QR-factorization for A, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

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i) $\vec{u_1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

$\vec{w_2} = \vec{v_2} - (\vec{u_1} \cdot \vec{v_2}) \vec{u_1}$

$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \left(\frac{1}{\sqrt{3}}\right) \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix}$

$\vec{u_2} = \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}} \vec{w_2} = \frac{1}{\frac{\sqrt{6}}{3}} \vec{w_2}$

$= \frac{3}{\sqrt{6}} \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}$

Basis formed b

$\boxed{\left\{ \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix} \right.}$

ii) $\boxed{\begin{pmatrix} 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 1/\sqrt{3} \\ 0 & \sqrt{6}/3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}}$

$\frac{1}{3} + \frac{1}{3}$

$\frac{1}{3} - \frac{1}{3}$

$\frac{1}{3} - \frac{1}{3}$

Problem 3. (4)

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(i) Decide if the equation $A\vec{x} = \vec{b}$ solvable, where $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(ii) Find the least-squares solution of the equation $A\vec{x} = \vec{b}$ for A and $\vec{b}$ in (i).

i) $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right]$ $\boxed{\text{Not Solvable}}$

System is inconsistent

$0 \neq -1$ $\quad rk(A) \neq rk(\hat{A})$

✓

ii) $\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$

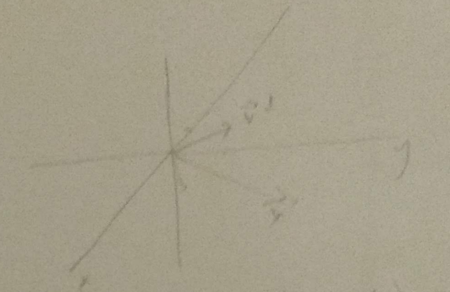
$(A^T A)^{-1} = \left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \right)^{-1} = \left(\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \right)^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$

$A^T \vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

✓

$\vec{x}^* = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}}$

$A(A^TA)^{-1}A^T$



Proj $\frac{(\vec{y}\cdot\vec{v})}{\|\vec{v}\|}\vec{v}$

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Problem 4. (4)

Let $V = \text{span}\{\vec{v_1}, \vec{v_2}\}$, where $\vec{v_1} = \begin{bmatrix}1\\1\\1\end{bmatrix}$, and $\vec{v_2} = \begin{bmatrix}0\\0\\1\end{bmatrix}$.

(i) Find a basis for the orthogonal complement $V^{\perp}$ of V.

(ii) Find the matrix for the projection to $V^{\perp}$, $Proj_{V^{\perp}} : \mathbf{R}^3 \to \mathbf{R}^3$ with respect to the standard basis of $\mathbf{R}^3$.

i) Find $\vec{x} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ s.t. $\vec{x}\cdot\vec{v_1} = 0$, $\vec{x}\cdot\vec{v_2} = 0$

$\begin{cases} 1x_1 + 1x_2 + 1x_3 = 0 \\ 0x_1 + 0x_2 + 1x_3 = 0 \end{cases}$

$x_3 = 0$

$x_1 + x_2 + 0 = 0$

$x_1 = -x_2$

let $t = x_2$

$\vec{x} = \begin{pmatrix}-t\\t\\0\end{pmatrix} = t\begin{pmatrix}-1\\1\\0\end{pmatrix}$

Basis formed by

$\boxed{\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}}$

✓

ii) 1. Find orthonormal basis of $V^{\perp}$

$\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix} = \begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}$

✓

2. Find projection matrix

$\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}\begin{pmatrix}-1/\sqrt{2} & 1/\sqrt{2} & 0\end{pmatrix} = \boxed{\begin{pmatrix}1/2 & -1/2 & 0\\-1/2 & 1/2 & 0\\0 & 0 & 0\end{pmatrix}}$

Problem 5. (4)

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$

(i) Find the eigenvalues of A and their algebraic multiplicities.

(ii) Find a basis for each eigenspace of A, and find the geometric multiplicity for each eigenvalue of A.

(iii) Decide if A is diagonalizable.

i) $\det \begin{pmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & -2 \\ 0 & 1 & 4-\lambda \end{pmatrix} = 0$

$(2-\lambda)(4-5\lambda+\lambda^2+2) = 0$

$(2-\lambda)(\lambda^2-5\lambda+6) = 0$

$(2-\lambda)(\lambda-2)(\lambda-3) = 0$

$(\lambda-2)^2(\lambda-3) = 0$

$\boxed{\lambda = 2 \quad \text{almu} = 2}$ $\boxed{\lambda = 3 \quad \text{almu} = 1}$

ii) $\begin{bmatrix} 2-2 & 1 & 1 & 0 \\ 0 & 1-2 & -2 & 0 \\ 0 & 1 & 4-2 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$

$x_2 + x_3 = 0$
$x_2 + 2x_3 = 0$ $\Rightarrow x_3 = 0$

$\begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} = t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Basis for E_2

$\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$

gemu = 1

$\begin{bmatrix} 2-3 & 1 & 1 & 0 \\ 0 & 1-3 & -2 & 0 \\ 0 & 1 & 4-3 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

$x_1 = 0$
$x_2 + x_3 = 0$
$x_2 = -x_3$

$\begin{pmatrix} 0 \\ -t \\ t \end{pmatrix} = t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

Basis for E_3

$\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$

gemu = 1

2

iii) $\boxed{A \text{ is not diagonalizable}}$

Dimensions of eigenspaces add up to 2, which is less than $n = 3$