to prepare for next test:
- practice row reduction
- practice matrix multiplication

1	2
2	2
3	0
4	4
5	3
T	11

MATH 33A Midterm I, Spring 2018

[redacted] 1A 1B,

Instruction: Justify all your answers. No points will be given without sufficient reasoning/calculations.

Problem 1. (4)

Given a linear system:

$$\left|\begin{array}{l} x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 + 3x_3 + x_4 = 1 \\ 2x_1 + 5x_2 + 8x_3 + 2x_4 = 2 \end{array}\right|$$

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(I) Find the coefficient matrix A, the augmented matrix $\tilde{A}$ and their reduced row-echelon form.
(II) Use the reduced row-echelon form of $\tilde{A}$ to determine if the system solvable. If it is, find all solutions of the system.

(I) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 5 & 8 & 2 \end{bmatrix}$

$\hat{A} = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 5 & 8 & 2 & 2 \end{array}\right] \begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{array}$

$\begin{array}{l} \\ -\mathrm{I} \rightarrow \\ 2\mathrm{I} \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 3 & 5 & 0 & 0 \end{array}\right]$

8-2=6 ...

$\begin{array}{l} \\ \rightarrow \\ -\mathrm{III} \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$

So this is 0
holy shit

rref$(\tilde{A})$

$\rightarrow \begin{array}{l} \mathrm{I}-\mathrm{II}-\mathrm{III} \\ 2\mathrm{III}-\mathrm{II} \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$

~~rref(A) =~~

(II) Yes, system is solvable since rank(A) = rank($\hat{A}$)

$x_1 + x_4 = 1 \Rightarrow x_1 = 1 - x_4$
$x_2 = 0$
$x_3 = 0$

let $x_4 = t \rightarrow \begin{bmatrix} 1-t \\ 0 \\ 0 \\ t \end{bmatrix}$ where $t \in \mathbb{R}$

Problem 2. (4)

Let

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 2 & 4 & 0 \\ 2 & 3 & 7 & 1 \end{bmatrix} \end{array} . \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array}$$

The matrix A defines a linear transformation $T: \mathbf{R}^4 \mapsto \mathbf{R}^3$ given by $\mathbf{y} = T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x} \in \mathbf{R}^4$

(I) Find a basis of the kernel of T.

(II) Find a basis of the image of T.

(I)

$$\begin{array}{c} \\ \text{II}-\text{I} \\ \text{III}-2\text{I} \end{array} \begin{vmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{vmatrix}$$

$$\rightarrow \begin{array}{c} \\ \\ \text{III}-\text{II} \end{array} \begin{vmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$x_1 + x_2 + 3x_3 + x_4 = 0$

$x_2 + x_3 - x_4 = 0$

$x_2 = x_4 - x_3$

$x_1 = -x_2 - 3x_3 - x_4$

$x_1 = x_3 - x_4 - 3x_3 - x_4$

$x_1 = -2x_3 - 2x_4$

Let $t = x_4$, $s = x_3$

$$\begin{bmatrix} t - s \\ -2s - 2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\text{Basis of } \ker(T): \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}}$$

(II) From ref(A), we know that

$$\begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

are redundant vectors

So $$\boxed{\text{basis of } \operatorname{im}(T): \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

②

Problem 3. (4)

Let matrices A and B be the following:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(I)Find $A \cdot B$.
(II) Find $(A \cdot B)^{-1}$.

Ans $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 7 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

(I) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1(1)+1(3)+1(1) & 1(1)+1(2)+1(1) & 1(0)+1(0)+1(1) \\ 1(1)+2(3)+1(1) & 1(1)+2(2)+1(1) & 1(0)+2(0)+1(1) \\ 0(1)+0(3)+1(1) & 0(1)+0(2)+1(1) & 0(0)+0(0)+1(1) \end{bmatrix}$

vector vector vector

$= \boxed{\begin{bmatrix} 5 & 4 & 1 \\ 8 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$

-2 oh shit it's row x col

$3(\frac{4}{9})$

(II) $\left[\begin{array}{ccc|ccc} 5 & 4 & 1 & 1 & 0 & 0 \\ 8 & 4 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} \to \left[\begin{array}{ccc|ccc} 1 & \frac{4}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{7}{4} & 0 & -\frac{1}{4} & 2 \\ 0 & 0 & 1 & -\frac{4}{9} & 9 & 12 \end{array}\right]$

$\frac{1}{5}$ I
III − II
IV − I

$\left[\begin{array}{ccc|ccc} 1 & \frac{4}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 4 & 7 & 0 & -1 & 8 \\ 0 & \frac{1}{5} & \frac{4}{5} & -\frac{1}{5} & 0 & 1 \end{array}\right] \to \begin{bmatrix} & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix}$

-2

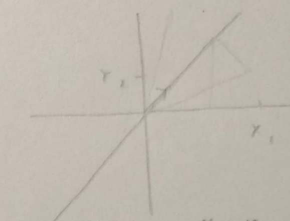
$\longrightarrow$
III − II

$\left[\begin{array}{ccc|ccc} 1 & \frac{4}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{7}{4} & 0 & -\frac{1}{4} & 2 \\ 0 & 0 & \frac{9}{4} & -1 & \frac{1}{4} & 3 \end{array}\right]$

0

$\vec{u} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$\left(\frac{1}{2} \right.$



Problem 4. (4)

Let L be the line in $\mathbf{R}^2$ defined by the equation $x_2 = x_1$, where (x_1, x_2) is the standard coordinate of $\mathbf{R}^2$.

(I) Find the matrices of the projection and reflection with respect to the line L in the standard basis of $\mathbf{R}^2$.
(II) Is any of these matrices invertible? If so, find its inverse.

(I)

$Ref = 2\,Proj - I$

$Ref + I = 2\,Proj$

$\frac{1}{2}(Ref + I) = Proj$

Reflection:
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Projection:
$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}}$$

(II) Reflection is invertible

inverse reflection = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

Projection is not invertible

4

Problem 5. (4)

Let

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

be a basis of $\mathbf{R}^3$.

3

(I) Let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find the coordinate $[\mathbf{v}]_B$ of $\mathbf{v}$ with respect to the basis B.

(II) A linear transformation T from $\mathbf{R}^3$ to $\mathbf{R}^3$ is given by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

Find the matrix for T with respect to the basis B.

(I) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 3 \end{array}\right]$

$c_1 = 3$

$c_1 + c_2 = 2$

$c_1 + c_2 + c_3 = 1$

$\Rightarrow c_1 = 3 \quad c_2 = -1 \quad c_3 = -1$

$\boxed{\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}}$ ✓

Find B^{-1}

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 & 0 & -1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & \end{array}\right]$

(II) $AB = BT$

$T = B^{-1}AB$

$T = B^{-1}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$= B^{-1}\begin{bmatrix} 1+1+1 & 1+1 & 1+0 \\ 0+2+3 & 0+2 & 0+0 \\ 0+0+4 & 0+0 & 0+0 \end{bmatrix} = B^{-1}\begin{bmatrix} 3 & 2 & 1 \\ 5 & 2 & 0 \\ 4 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 5 & 2 & 0 \\ 4 & 0 & 0 \end{bmatrix}$ -1

$= \begin{bmatrix} 0+0+1 & 0+0+5 & 0 \\ 0+2+(-1) & 0+2-2 & 0 \\ 1+(-1)(1)+(-1) & 5+0+0 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 5 & 0 \\ \dots \end{bmatrix}$