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MATH 33A Midterm II, Spring 2018

**Name:**

**Circle**
**Castle**

**Instruction: Justify all your answers. No points will be given without sufficient reasoning/calculations.**

**Problem 1.** (4)

(i) Use row or column reduction to compute $detA$, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 1 & 4 & 5 & 7 \end{bmatrix}.$$

(ii) Find $detB$, where

$$B = \begin{bmatrix} 0 & 0 & a & 4 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & b & c \\ 1 & d & e & f \end{bmatrix}.$$

Here $a$, $b$, $c$, $d$, $e$ and $f$ are constants.

(iii) Find $det(AB)$, $det(A^{-1})$ and $det(B^{-1})$.

4

i) $A = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 1 & 4 & 5 & 7 \end{vmatrix} \xrightarrow[-I+IV]{-I+II,\ -I+III} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 4 \\ 0 & 3 & 4 & 6 \end{vmatrix} \xrightarrow[-3II+IV]{-II+I,\ -2II+III} \begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & -3 \end{vmatrix} \xrightarrow[-1\ IV]{-1\cdot III} \begin{vmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 3 \end{vmatrix}$

$\xrightarrow[-2III+IV]{III+I,\ -2III+II} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{vmatrix}$

$det(A) = (-1)^2(1\cdot 1\cdot 1\cdot -1) = -1$ $\boxed{det(A) = -1}$

ii) $B = \begin{bmatrix} 0 & 0 & a & 4 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & b & c \\ 1 & d & e & f \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & B \\ A^1 & C \end{bmatrix} = det\,A\,det\,B = (-2)(-12) = 24$

check. $\begin{bmatrix} 0 & 0 & a & 4 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & b & c \\ 1 & d & e & f \end{bmatrix} \xrightarrow{\frac{1}{3}II} \begin{bmatrix} 0 & 0 & a & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & b & c \\ 1 & d & e & f \end{bmatrix} \xrightarrow{-aII+I} \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & b & c \\ 1 & d & e & f \end{bmatrix}$ $det\,B = 3(1\cdot 2\cdot 1\cdot 4)$
$= 24$

$\boxed{det\,B = 24}$

iii) $\det(AB) = \det(A)\det(B) = (-1)(24) = -24$

$$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{-1} = -1$$

$$\det(B^{-1}) = \frac{1}{\det(B)} = \frac{1}{24}$$

**Problem 2.** (4)

(4)

Let $V = \text{span}\{\vec{v_1}, \vec{v_2}\}$ be a subspace in $\mathbf{R}^3$, where $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(i) Find an orthonormal basis for $V$ by using Gram-Schmidt process.

(ii) Find the QR-factorization for $A$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

i) $\vec{u}_1 = \dfrac{\vec{V}_1}{\|\vec{V}_1\|}$ $\quad \vec{V}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \|\vec{V}_1\| = \sqrt{1+1+1} = \sqrt{3}$

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{V}_2 - (\vec{u}_1 \cdot \vec{V}_2)\vec{u}_1}{\|\vec{V}_2 - (\vec{u}_1 \cdot \vec{V}_2)\vec{u}_1\|}$$

$$\vec{u}_1 \cdot \vec{v}_2 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1/\sqrt{3}$$

$$(\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = 1/\sqrt{3} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\vec{u}_2 = \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix} \cdot \frac{3}{\sqrt{6}}$$

$$\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix}$$

$$\vec{u}_2 = \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

$$\|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}} = \frac{\sqrt{6}}{3}$$

$$\boxed{\text{orthonormal basis for } V: \left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \right\}}$$

ii) Since $Q = \begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$ then $R = Q^T A$ where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$

2

$$Q^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}$$

$$R = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{3} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{3} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} \end{bmatrix}$$

$\longrightarrow$

So QR Factorization:

$$\underbrace{\begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \sqrt{3} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} \end{bmatrix}}_{R}$$

**Problem 3.** (4)

(i) Decide if the equation $A\vec{x} = \vec{b}$ solvable, where $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

4

(ii) Find the least-squares solution of the equation $A\vec{x} = \vec{b}$ for $A$ and $\vec{b}$ in (i).

i) $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow[-I+III]{-I+II} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{switch } II \text{ und } III} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right]$ ✓

Since the last row suggests that $0=-1$, this equation $A\vec{x}=\vec{b}$ is <u>not</u> solvable for $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

ii) $\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$ $\quad A^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

$(A^TA)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$ $\quad A^TA = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$

$\vec{x}^* = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ ✓

least-squares solution: $\vec{x}^* = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

3

2

**Problem 4.** (4)

Let $V = \text{span}\,\{\vec{v_1}, \vec{v_2}\}$, where $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(i) Find a basis for the orthogonal complement $V^\perp$ of $V$.

(ii) Find the matrix for the projection to $V^\perp$, $Proj_{V^\perp} : \mathbf{R}^3 \to \mathbf{R}^3$ with respect to the standard basis of $\mathbf{R}^3$.

i) $\vec{u}_1 = \dfrac{\vec{V}_1}{\|\vec{V}_1\|}$ $\quad \vec{V}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \|\vec{V}_1\| = \sqrt{1+1+1} = \sqrt{3}$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\vec{u}_1 \cdot \vec{V}_2 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1/\sqrt{3}$$

$$\vec{u}_2 = \frac{\vec{V}_2 - (\vec{u}_1 \cdot \vec{V}_2)\vec{u}_1}{\|\vec{V}_2 - (\vec{u}_1 \cdot \vec{V}_2)\vec{u}_1\|}$$

$$(\vec{u}_1 \cdot \vec{V}_2)\vec{u}_1 = 1/\sqrt{3} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\vec{V}_2 - (\vec{u}_1 \cdot \vec{V}_2)\vec{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$\|\vec{V}_2 - (\vec{u}_1 \cdot \vec{V}_2)\vec{u}_1\| = \sqrt{\frac{1}{9} + \frac{1}{9} + \frac{4}{9}} = \frac{\sqrt{6}}{3}$$

$$\vec{u}_2 = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix} \cdot \frac{3}{\sqrt{6}}$$

$$\vec{u}_2 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$$\boxed{\text{Basis}: \left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}}$$

$\frac{1}{3} + \frac{1}{2}$

ii) $Proj_{V^\perp} = AA^T = B$ $\quad B = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$

Consistent so ok

(+2)

$$\boxed{B = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

**Problem 5.** (4)

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$

(i) Find the eigenvalues of $A$ and their algebraic multiplicities.

(ii) Find a basis for each eigenspace of $A$, and find the geometric multiplicity for each eigenvalue of $A$.

(iii) Decide if $A$ is diagonalizable.

i) $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$ $\quad A-\lambda I_3 = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & -2 \\ 0 & 1 & 4-\lambda \end{bmatrix}$ Solve $\det(A-\lambda I_3)=0$

$\det(A-\lambda I_3) = ((1-\lambda)(4-\lambda)-(-2))(2-\lambda) - 0 + 0 = [4-5\lambda+\lambda^2+2](2-\lambda)$

$= [\lambda^2-5\lambda+6](2-\lambda) = (\lambda-2)(\lambda-3)(2-\lambda) = 0$

$\boxed{\begin{array}{l} \lambda=2, \text{ algebraic multiplicity of } 2 \\ \lambda=3, \text{ algebraic multiplicity of } 1 \end{array}}$

ii) For $\lambda=2$: $A-2I_3=0 \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow[-I+III]{I+II} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ so $\boxed{\lambda=2 \text{ has geometric multiplicity of } 1}$

$\boxed{\text{basis}: \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$

5

For $\lambda=3$, $A-3I_3=0 \Rightarrow \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow[\text{switch } II \,\&\, III]{-1 \cdot I} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right] \xrightarrow[2II+III]{II+I}$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $\boxed{\begin{array}{l} \text{so } \lambda=3 \text{ has geometric multiplicity of } 1 \\ \text{basis}: \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \end{array}}$

4

iii) Since $\sum$ algebraic multiplicities $\neq \sum$ geometric multiplicities

$$3 \neq 2$$

then A is <u>not</u> diagonalizable