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MATH 33A Midterm I, Spring 2018

Name

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Instruction: Justify all your answers. No points will be given without sufficient reasoning/calculations.

Problem 1. (4)

Given a linear system:

$$\left|\begin{array}{l} x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 + 3x_3 + x_4 = 1 \\ 2x_1 + 5x_2 + 8x_3 + 2x_4 = 2 \end{array}\right|$$

(I) Find the coefficient matrix A, the augmented matrix $\tilde{A}$ and their reduced row-echelon form.
(II) Use the reduced row-echelon form of $\tilde{A}$ to determine if the system solvable. If it is, find all solutions of the system.

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1) coefficient matrix $A = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 5 & 8 & 2 \end{vmatrix}$

Augmented matrix $\hat{A} = \left|\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 5 & 8 & 2 & 2 \end{array}\right|$

$rref(\hat{A}) = \left|\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 5 & 8 & 2 & 2 \end{array}\right| \xrightarrow[-2(I)+(III)]{-(I)+(II)} \left|\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 3 & 6 & 0 & 0 \end{array}\right| \; III \cdot \frac{1}{3}$

$\rightarrow \left|\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right| \xrightarrow[-(II)+III]{-(II)+I} \left|\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right|$

$\boxed{rref(\hat{A}) = \left|\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right|}$

$\Rightarrow$

ii) According to the rref, the system is solvable

$$\left|\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right| \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - x_4 + x_3 \\ -2x_3 \\ x_3 \\ x_4 \end{bmatrix}$$

So for any arbitrary value $x_3 = a$ and $x_4 = b$, the solution space takes the form $\begin{bmatrix} 1 - b + a \\ -2a \\ a \\ b \end{bmatrix}$

and the system has infinite solutions

Problem 2. (4)

Let

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 2 & 4 & 0 \\ 2 & 3 & 7 & 1 \end{bmatrix}.$$

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The matrix A defines a linear transformation $T: \mathbf{R}^4 \mapsto \mathbf{R}^3$ given by $\mathbf{y} = T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x} \in \mathbf{R}^4$

(I) Find a basis of the kernel of T.

(II) Find a basis of the image of T.

1) Find rref(A) when $A\vec{x} = 0$ (Ker(A))

$$\text{rref}(A) = \left|\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 \\ 2 & 3 & 7 & 1 & 0 \end{array}\right| \xrightarrow[-2(I)+(III)]{-(I)+(II)} \left|\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right| \xrightarrow[-(II)+(I)]{-(II)+(III)}$$

$$\text{rref}(A) = \left|\begin{array}{cccc|c} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right|$$

so solution space takes the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 - 2x_4 \\ x_4 - x_3 \\ x_3 \\ x_4 \end{bmatrix}$$

and

$$\begin{bmatrix} -2x_3 - 2x_4 \\ x_4 - x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$\boxed{\text{So the basis of Ker}(T) \text{ is composed of the vectors } \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}}$

ii) Since $\text{rref}(A) = \left|\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 2 & 0 \\ 0 & \boxed{1} & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right|$ 2 and it contains leading 1's in the first and second column, the basis of Im(T) is composed of the corresponding 1st and 2nd column vectors in A.

$\boxed{\text{Basis of Im}(T) \text{ composed of } \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$

Problem 3. (4)

Let matrices A and B be the following:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(I)Find $A \cdot B$.
(II) Find $(A \cdot B)^{-1}$.

I)

$$A \cdot B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+1+0 & 3+2+0 & 1+1+1 \\ 1+2+0 & 3+4+0 & 1+2+1 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$\boxed{A \cdot B = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 7 & 4 \\ 0 & 0 & 1 \end{bmatrix}}$$

II) To find $(A \cdot B)^{-1}$, create augmented matrix: $[A \cdot B \,|\, I_n]$ and find rref.

$-\frac{3}{2}$ 3

$\frac{9}{2} + \frac{8}{2}$

$\frac{15}{2} + \frac{14}{2} = \frac{1}{2}$

$$\left[\begin{array}{ccc|ccc} 2 & 5 & 3 & 1 & 0 & 0 \\ 3 & 7 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(I) \cdot \frac{1}{2}} \left[\begin{array}{ccc|ccc} 1 & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 3 & 7 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-3(I) + (II)}$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2 \cdot (II)} \left[\begin{array}{ccc|ccc} 1 & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-\frac{5}{2}(II) + (I)}$$

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$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -7 & 5 & 0 \\ 0 & 1 & 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{(III)}+\text{(I)}]{-\text{(III)}+\text{(II)}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 5 & 1 \\ 0 & 1 & 0 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Since rref yields $[I_n \mid (A\cdot B)^{-1}]$, then

$$\boxed{(A\cdot B)^{-1} = \begin{bmatrix} -7 & 5 & 1 \\ 3 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$$

Check. $(A\cdot B)^{-1} = B^{-1} A^{-1}$

$$A^{-1} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ & & & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^{-1} = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 4 & -2 & 3 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 & 3 & -4 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 & -4 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 & -4 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 3 & -2 \\ 3 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 4. (4)

Let L be the line in $\mathbf{R}^2$ defined by the equation $x_2 = x_1$, where (x_1, x_2) is the standard coordinate of $\mathbf{R}^2$.

(I) Find the matrices of the projection and reflection with respect to the line L in the standard basis of $\mathbf{R}^2$.
(II) Is any of these matrices invertible? If so, find its inverse.

1) Since we are in standard basis and $x_2 = x_1$, the vector $\vec{v}$ along L that is being projected onto can be written as $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

projection matrix $= \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix} = \frac{1}{1^2+1^2} \begin{bmatrix} 1^2 & 1 \cdot 1 \\ 1 \cdot 1 & 1^2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\boxed{\text{projection matrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}}$

reflection matrix $= 2(\text{projection matrix}) - I_2 = 2 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\boxed{\text{reflection matrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$

ii) Since the determinant of the projection matrix $= 0$ $\left(\frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} = 0\right)$, $\boxed{\text{projection matrix is \underline{not} invertible.}}$

The reflection matrix determinant $\neq 0$, so it would be invertible. Check by rref on [reflection matrix | I_2]

$$\left|\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right| \xrightarrow{\text{switch (I), (II)}} \left|\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right|$$

So $\boxed{\text{inverse of reflection matrix is } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$

Problem 5. (4)

Let

$$\mathcal{B} = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}$$

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be a basis of $\mathbf{R}^3$.

(I) Let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find the coordinate $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v}$ with respect to the basis $\mathcal{B}$.

(II) A linear transformation T from $\mathbf{R}^3$ to $\mathbf{R}^3$ is given by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

Find the matrix for T with respect to the basis $\mathcal{B}$.

I) there exists some coordinate $[\vec{v}]_B = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ such that

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + v_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

so $\begin{cases} v_1 + v_2 + v_3 = 1 \\ v_1 + v_2 = 2 \\ v_1 = 3 \end{cases}$ which is rather trivial and yiels $v_1 = 3,\ v_2 = -1,\ v_3 = -1$

so coordinate $\boxed{[\vec{v}]_B = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}}$ ✓

II) We can use the relationship $AS = SB$ with

$A = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{vmatrix}$ and $S = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$ (as denoted in part 1)

so $B = S^{-1}AS$. First we need to find S^{-1}

⟹

$$\left|\begin{array}{ccc|ccc} 1&1&1&1&0&0\\ 1&1&0&0&1&0\\ 1&0&0&0&0&1 \end{array}\right| \xrightarrow[-(\mathrm{I})+(\mathrm{III})]{-(\mathrm{I})+(\mathrm{II})} \left|\begin{array}{ccc|ccc} 1&1&1&1&0&0\\ 0&0&-1&-1&1&0\\ 0&-1&-1&-1&0&1 \end{array}\right| \xrightarrow[-1(\mathrm{III})]{\text{switch } (\mathrm{II}),(\mathrm{III}) \quad -1(\mathrm{II})}$$

$$\left|\begin{array}{ccc|ccc} 1&1&1&1&0&0\\ 0&1&1&1&0&-1\\ 0&0&1&1&-1&0 \end{array}\right| \xrightarrow{-(\mathrm{II})+(\mathrm{I})} \left|\begin{array}{ccc|ccc} 1&0&0&0&0&1\\ 0&1&1&1&0&-1\\ 0&0&1&1&-1&0 \end{array}\right| \xrightarrow{-(\mathrm{III})+(\mathrm{II})}$$

$$\left|\begin{array}{ccc|ccc} 1&0&0&0&0&1\\ 0&1&0&0&1&-1\\ 0&0&1&1&-1&0 \end{array}\right| \quad \text{so } S^{-1} = \begin{bmatrix} 0&0&1\\ 0&1&-1\\ 1&-1&0 \end{bmatrix}$$

$$B = S^{-1}AS = S^{-1}\begin{bmatrix} 1&1&1\\ 0&2&3\\ 0&0&4 \end{bmatrix}\begin{bmatrix} 1&1&1\\ 1&1&0\\ 1&0&0 \end{bmatrix} = S^{-1}\begin{bmatrix} 1+1+1 & 1+1+0 & 1+0+0\\ 0+2+3 & 0+2+0 & 0+0+0\\ 0+0+4 & 0+0+0 & 0+0+0 \end{bmatrix}$$

$$= S^{-1}\begin{bmatrix} 3&2&1\\ 5&2&0\\ 4&0&0 \end{bmatrix} = \begin{bmatrix} 0&0&1\\ 0&1&-1\\ 1&-1&0 \end{bmatrix}\begin{bmatrix} 3&2&1\\ 5&2&0\\ 4&0&0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+4 & 0+0+0 & 0+0+0\\ 0+\ -4 & 0+2+0 & 0+0+0\\ 3-5+0 & 2-2+0 & 1+0+0 \end{bmatrix} = \begin{bmatrix} 4&0&0\\ 1&2&0\\ -2&0&1 \end{bmatrix}$$

so $\boxed{B = \begin{bmatrix} 4&0&0\\ 1&2&0\\ -2&0&1 \end{bmatrix}}$ ✓