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2	4
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MATH 33A Midterm I, Spring 2018

Name:

Circle Your TA's Name and Section Number: Breen 1A 1B, Castle 1E 1F, Qiu 1C 1D

Instruction: Justify all your answers. No points will be given without sufficient reasoning/calculations.

Problem 1. (4)

Given a linear system:

$$\left| \begin{array}{l} x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 + 3x_3 + x_4 = 1 \\ 2x_1 + 5x_2 + 8x_3 + 2x_4 = 2 \end{array} \right|$$

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(I) Find the coefficient matrix A, the augmented matrix $\tilde{A}$ and their reduced row-echelon form.
(II) Use the reduced row-echelon form of $\tilde{A}$ to determine if the system solvable. If it is, find all solutions of the system.

(I) $\boxed{A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 5 & 8 & 2 \end{bmatrix}. \quad \tilde{A} = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 5 & 8 & 2 & 2 \end{array}\right]}$

$\tilde{A} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\boxed{\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad \text{rref}(\tilde{A}) = \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].}$

(II) rank $(A) = 2 =$ rank $(\tilde{A}) \Rightarrow$ system is $\boxed{\text{solvable}}$

$$\left| \begin{array}{l} x_1 - x_3 + x_4 = 1 \\ x_2 + 2x_3 = 0 \end{array} \right| \Rightarrow \left| \begin{array}{l} x_1 = x_3 - x_4 + 1 \\ x_2 = -2x_3 \end{array} \right|.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \boxed{\left\{ \begin{bmatrix} x_3 - x_4 + 1 \\ -2x_3 \\ x_3 \\ x_4 \end{bmatrix} \middle| \; x_3 \in \mathbb{R}, \; x_4 \in \mathbb{R} \right\}}$$

Problem 2. (4)

Let

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 2 & 4 & 0 \\ 2 & 3 & 7 & 1 \end{bmatrix}.$$

The matrix A defines a linear transformation $T : \mathbf{R}^4 \mapsto \mathbf{R}^3$ given by $\mathbf{y} = T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x} \in \mathbf{R}^4$

(I) Find a basis of the kernel of T.

(II) Find a basis of the image of T.

(I) $\begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 2 & 4 & 0 \\ 2 & 3 & 7 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = rref(A)$

$x_1 = -2x_3 - 2x_4$

$x_2 = -x_3 + x_4$

$x_3 = x_3$

$x_4 = x_4.$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

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A basis for ker(T) is $\boxed{\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}}$

(II) Because there is a pivot in columns 1 and 2 of rref(A), we choose columns 1 and 2 in A as ~~a~~ basis for image(T).

A basis for image(T) is $\boxed{\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}}$

Problem 3. (4)

Let matrices A and B be the following:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(I)Find $A \cdot B$.
(II) Find $(A \cdot B)^{-1}$.

(I) $AB = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 5 & 3 \\ 3 & 7 & 4 \\ 0 & 0 & 1 \end{bmatrix}}$

(II) $\left[AB \mid I_3 \right] \Rightarrow \left[I_3 \mid (AB)^{-1} \right]$

$$\left[\begin{array}{ccc|ccc} 2 & 5 & 3 & 1 & 0 & 0 \\ 3 & 7 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 2 & 5 & 3 & 1 & 0 & 0 \\ 1 & 2 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 3 & -2 & 0 \\ 1 & 2 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 5 & 1 \\ 0 & 1 & 0 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

$(AB)^{-1} = \boxed{\begin{bmatrix} -7 & 5 & 1 \\ 3 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$

Problem 4. (4)

Let L be the line in $\mathbf{R}^2$ defined by the equation $x_2 = x_1$, where (x_1, x_2) is the standard coordinate of $\mathbf{R}^2$.

(I) Find the matrices of the projection and reflection with respect to the line L in the standard basis of $\mathbf{R}^2$.
(II) Is any of these matrices invertible? If so, find its inverse.

(I) Let $\vec{u} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ $L = \text{span}(\vec{u})$ $\|\vec{u}\| = 1$.

Let $P = \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_2 u_1 & u_2^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$\text{proj}_L \vec{x} = \boxed{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}} \vec{x} = P\vec{x}.$

$\text{ref}_L \vec{x} = (2P - I_2)\vec{x} = R\vec{x}$

$R = 2P - I_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$

(II) $\boxed{P \text{ is not invertible}}$ since $\text{proj}_L \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \text{proj}_L \begin{bmatrix} 1 \\ -1 \end{bmatrix}$,

yet $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\boxed{R \text{ is invertible}}$ since geometrically the reflection of the reflection of a vector over the same line is always that vector.

$R^{-1} = R = \boxed{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$

Problem 5. (4)

Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

be a basis of $\mathbf{R}^3$.

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(I) Let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find the coordinate $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v}$ with respect to the basis $\mathcal{B}$.

(II) A linear transformation T from $\mathbf{R}^3$ to $\mathbf{R}^3$ is given by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}.$$

Find the matrix for T with respect to the basis $\mathcal{B}$.

(I) $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$[\vec{v}]_{\mathcal{B}} = \boxed{\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}}$

(II) Let $S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. We first find S^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]. \quad S^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

Let B be the $\mathcal{B}$-matrix of T.

$$AS = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 5 & 2 & 0 \\ 4 & 0 & 0 \end{bmatrix}.$$

$$B = S^{-1}AS = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 5 & 2 & 0 \\ 4 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 4 & 0 & 0 \\ 1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}}$$