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1. (10 points) (a) Solve the following system of linear equations:

$$\begin{vmatrix} 2x - 2y + 22z = 40 \\ -x + 5y - 15z = -8 \\ -x + y - 7z = -12 \end{vmatrix}$$

(b) How many solutions does the system

$$\begin{vmatrix} 2x - 2y + 22z = 0 \\ -x + 5y - 15z = 0 \\ -x + y - 7z = 0 \end{vmatrix}$$

have?

(c) For each of the following statements, circle T for True, F for False.

✓ T F The matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

is in reduced row echelon form.

✓ T F If A^2 is an invertible matrix then so is A^3.

✓ T F There exists a 2×2 matrix A such that $A^2 - A = I_2$.

(T) F There exists a 2×2 matrix A such that $A^2 - A = I_2$.

a) $\left[\begin{array}{ccc|c} 2 & -2 & 22 & 40 \\ -1 & 5 & -15 & -8 \\ -1 & 1 & -7 & -12 \end{array}\right] \begin{array}{l} \div 2 \\ +\frac{1}{2}(I) \\ +\frac{1}{2}(I) \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 11 & 20 \\ 0 & 4 & -4 & 12 \\ 0 & 0 & 4 & 8 \end{array}\right] \begin{array}{l} +\frac{1}{4}(II) \\ \div 4 \\ \div 4 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 10 & 23 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} -10(III) \\ +(III) \\ \end{array}$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$ $\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}}$ ✓

b) $\boxed{\text{one}}$ (since the rank of the coeff matrix is just 3) 3×3

✓

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

A^2 $(AA)^{-1}$ A^3 $(AAA)^{-1}$

$A^{-1}A^{-1}$ $A^{-1}A^{-1}A^{-1}$

$A(A-I) = I$

2. (10 points) Let $T, U : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformations satisfying

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$$T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}, \qquad T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\1\end{bmatrix}$$

and

$$U\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}, \qquad U\left(\begin{bmatrix}1\\3\end{bmatrix}\right) = \begin{bmatrix}-1\\3\end{bmatrix}$$

(a) Find the representing matrix A of T, and the representing matrix B of U.

(b) Compute the products AB and BA.

(c) Describe the transformation represented by $BA - AB$ geometrically.

a) Let $T = [\vec{v_1}\ \vec{v_2}]$ with $\vec{v_1}, \vec{v_2}$ being column vectors

$[\vec{v_1}\ v_2]\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$ $\qquad$ $[\vec{v_1}\ \vec{v_2}]\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$

$2\vec{v_1} + \vec{v_2} = \begin{bmatrix}2\\1\end{bmatrix}$ $\qquad$ $\vec{v_1} + \vec{v_2} = \begin{bmatrix}-1\\1\end{bmatrix}$

$\vec{v_1} + (\vec{v_1} + \vec{v_2}) = \begin{bmatrix}2\\1\end{bmatrix}$ $\qquad$ $\vec{v_2} = \begin{bmatrix}-1\\1\end{bmatrix} - \vec{v_1} = \begin{bmatrix}-1\\1\end{bmatrix} - \begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}-4\\1\end{bmatrix}$

$\vec{v_1} + \begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$ $\qquad$ $\boxed{A = \begin{bmatrix}3 & -4\\0 & 1\end{bmatrix}}$ ✓

$\vec{v_1} = \begin{bmatrix}3\\0\end{bmatrix}$

$\vec{v_1} + (\vec{v_1} + \vec{v_2}) = \begin{bmatrix}2\\1\end{bmatrix}$ $\qquad$ $\vec{v_2} = \begin{bmatrix}-1\\1\end{bmatrix} - \vec{v_1} = \begin{bmatrix}-1\\1\end{bmatrix} - \begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}-4\\1\end{bmatrix}$

$\vec{v_1} + \begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$ $\qquad$ $\boxed{A = \begin{bmatrix}3 & -4\\0 & 1\end{bmatrix}}$ ✓

$\vec{v_1} = \begin{bmatrix}3\\0\end{bmatrix}$

$U = [\vec{w_1}\ \vec{w_2}]$ with $\vec{w_1}, \vec{w_2}$ column vectors

$[\vec{w_1}\ \vec{w_2}]\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$ $\qquad$ $[\vec{w_1}\ \vec{w_2}]\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}-1\\3\end{bmatrix}$

$0 + \vec{w_2} = \begin{bmatrix}1\\1\end{bmatrix}$ $\qquad$ ✓ $\qquad$ $\vec{w_1} + 3\vec{w_2} = \begin{bmatrix}-1\\3\end{bmatrix}$ $\qquad$ $\vec{w_1} = \begin{bmatrix}-1\\3\end{bmatrix} - 3\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-4\\0\end{bmatrix}$

$\boxed{B = \begin{bmatrix}-4 & 1\\0 & 1\end{bmatrix}}$ $\qquad$ $\begin{bmatrix}0 & 16\\0 & 0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}16\\0\end{bmatrix}$

b) $AB = \begin{bmatrix}3 & -4\\0 & 1\end{bmatrix}\begin{bmatrix}-4 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}-12-0 & 3-4\\0+0 & 0+1\end{bmatrix} = \boxed{\begin{bmatrix}-12 & -1\\0 & 1\end{bmatrix}}$

$BA = \begin{bmatrix}-4 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}3 & -4\\0 & 1\end{bmatrix} = \begin{bmatrix}-12+0 & 16+1\\0+0 & 0+1\end{bmatrix} = \boxed{\cancel{\begin{bmatrix}-12 & 15\\0 & 1\end{bmatrix}}}$ −1

c) $BA - AB = \begin{bmatrix}0 & 16\\0 & 0\end{bmatrix}$ unit projection onto x axis, scaling by y component times 16 −1

+ rotation

3. (10 points) (a) Identify the redundant vectors in the following list:

$$\begin{array}{cccccc} \vec{v}_1 & \text{1st} & \text{2nd} & \vec{v}_2 & \text{3rd} & \\ \begin{bmatrix}1\\2\\3\end{bmatrix}, & \begin{bmatrix}0\\0\\0\end{bmatrix}, & \begin{bmatrix}-1\\-2\\-3\end{bmatrix}, & \begin{bmatrix}1\\1\\1\end{bmatrix}, & \begin{bmatrix}0\\2\\4\end{bmatrix}, & \begin{bmatrix}0\\0\\1\end{bmatrix} \end{array}$$

1st: $0\vec{v}_1$

2nd: $-\vec{v}_1$

3rd: $2\vec{v}_1 - 2\vec{v}_2$

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others not?

(b) For each of the following statements, circle T for True, F for False.

T F Given two subspaces V and W of $\mathbb{R}^n$, the set of vectors

$$\vec{v} + \vec{w},$$

where $\vec{v} \in V$, $\vec{w} \in W$, is a subspace of $\mathbb{R}^n$.

T F There exists a 3×2 matrix whose image is $\mathbb{R}^3$.

T F If a subspace W of $\mathbb{R}^2$ contains neither $\begin{bmatrix}1\\0\end{bmatrix}$ nor $\begin{bmatrix}1\\1\end{bmatrix}$, then W consists of the zero vector only.

T F Given an $n \times n$ matrix A, the intersection

$$\ker(A) \cap \text{image}(A)$$

consists of the zero vector only.

T? T F If an $n \times m$ matrix A has rank m then the column vectors of A are linearly independent.

T F For every subspace V of $\mathbb{R}^3$ there exists a 3×3 matrix A such that image$(A) = V$.

4. (10 points) (a) Let L be a line in $\mathbb{R}^2$ spanned by a non-zero vector $\vec{v}$.

For each of the following transformations in the plane, determine the kernel and the image. If the transformation is invertible, also describe the inverse transformation geometrically.

1. proj_L followed by a scaling by a factor of 2;
2. proj_L followed by $\text{proj}_{L^\perp}$ (recall that $L^\perp$ is the orthogonal complement to L);
3. ref_L followed by a rotation through 180°;
4. $A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

(b) Let A be an $n \times n$ matrix and assume that $A^{2016} = 0$. Show that the matrix $I_n - A$ is invertible.

a) ① The kernel is the span of any vector $\perp$ to $\vec{v}$ ✓

The image is span$(\vec{v})$ ✓

② The kernel is any vector in $\mathbb{R}^2$ ✓

The image is the zero vector ✓

The image is the zero vector

(3.) The kernel is the zero vector : $\{\vec{0}\}$

The image is any vector in $\mathbb{R}^2$

The inverse transformation is a 180° rotation followed by ~~an~~ ref_L

(4.) $\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ~~Ker(A) = span{[ ]}~~

$x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\boxed{\ker(A) = \text{span}\begin{bmatrix} -3 \\ 1 \end{bmatrix}}$

$x_1 + 3x_2 = 0$ $\quad 0x_1 + 0x_2 = 0$ $\boxed{\text{im}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)}$

$x_1 = -3x_2$ $\quad x_2 = t, \quad x_1 = -3t$

$-3t$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$

b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

If $A^{2016} = 0$ then $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

which is invertible (generalize for $n \times n$)

rank(I-A)=n