Math 33A/1
Spring 2016
04/22/16
Time Limit: 50 Minutes

Name (Print): ______________________
SID Number: ______________________

Day \ T.A.	David	Casey	Adam
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

This exam contains 6 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, cross the box correspondiing to your discussion section, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Use a pen to record your answers. Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. (10 points) Consider the following system of linear equations:

$$\begin{vmatrix} 2x - 4y + z = 0 \\ x + ky = 0 \\ 2y + kz = 1 \end{vmatrix},$$

where k is a real constant.

(a) For which values of k does the system have a unique solution? No solutions? Infinitely many?

(b) Solve the system when $k = 0$.

Solution:

(a) We write down the corresponding augmented matrix and perform row-reduction. Notice that for example in the third step we are not allowed to divide the second row by $-4k - 2$ since we don't know whether this is 0 or not. You can't divide by zero!

$$\begin{bmatrix} 2 & -4 & 1 & 0 \\ 1 & k & 0 & 0 \\ 0 & 2 & k & 1 \end{bmatrix} \to \begin{bmatrix} 1 & k & 0 & 0 \\ 2 & -4 & 1 & 0 \\ 0 & 2 & k & 1 \end{bmatrix} \to \begin{bmatrix} 1 & k & 0 & 0 \\ 0 & -4-2k & 1 & 0 \\ 0 & 2 & k & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & k & 0 & 0 \\ 0 & 2 & k & 1 \\ 0 & -4-2k & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & k & 0 & 0 \\ 0 & 1 & \frac{k}{2} & \frac{1}{2} \\ 0 & -4-2k & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & k & 0 & 0 \\ 0 & 1 & \frac{k}{2} & \frac{1}{2} \\ 0 & 0 & 1 + (4+2k)\frac{k}{2} & \frac{4+2k}{2} \end{bmatrix}$$

Now, we distinguish two cases:

- Suppose the third entry in the third row is 0, i.e.

$$0 = 1 + (4+2k)\frac{k}{2} = 1 + 2k + k^2 = (k+1)^2,$$

  i.e. $k = -1$. In that case the last row becomes $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$ and the system is thus inconsistent.

- Suppose the third entry in the third row is not zero, i.e. by our analysis above $k \neq -1$. In that case the rank of the coefficient matrix is 3 and there is a unique solution to the system.

Upshot: There is a unique solution if $k \neq -1$ and no solutions if $k = -1$.

(b) Plugging $k = 0$ into the last matrix we obtained in part (a) we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

hence the system has the unique solution $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 2 \end{bmatrix}$.

2. (10 points) (a) Find the inverse of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix}$.

(b) Find a 4×3 matrix A satisfying

$$A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \qquad A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \end{bmatrix}, \qquad A\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ -1 \\ 1 \end{bmatrix}.$$

(Hint: Notice that e.g. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.)

Solution:

(a) We write down the matrix alongside the identity matrix and perform row-reduction.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right]$$

Thus we see that the inverse is

$$\begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -2 & 1 & 0 \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

(b) We have that

$$A\begin{bmatrix}1\\0\\0\end{bmatrix} = A\left(2\begin{bmatrix}1\\1\\1\end{bmatrix} - 2\begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\2\end{bmatrix}\right)$$
$$= 2A\begin{bmatrix}1\\1\\1\end{bmatrix} - 2A\begin{bmatrix}0\\1\\0\end{bmatrix} - A\begin{bmatrix}1\\0\\2\end{bmatrix}$$
$$= 2\begin{bmatrix}1\\2\\0\\4\end{bmatrix} - 2\begin{bmatrix}0\\1\\-1\\-2\end{bmatrix} - \begin{bmatrix}-3\\2\\-1\\1\end{bmatrix}$$
$$= \begin{bmatrix}5\\0\\3\\11\end{bmatrix}$$

And similarly,

$$A\begin{bmatrix}0\\0\\1\end{bmatrix} = A\left(\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\0\\4\end{bmatrix} - \begin{bmatrix}5\\0\\3\\11\end{bmatrix} - \begin{bmatrix}0\\1\\-1\\-2\end{bmatrix} = \begin{bmatrix}-4\\1\\-2\\-5\end{bmatrix}$$

So

$$A = \begin{bmatrix}5 & 0 & -4\\0 & 1 & 1\\3 & -1 & -2\\11 & -2 & -5\end{bmatrix}$$

3. (10 points) (a) Find the matrix of reflection about the line $4y = 3x$ in $\mathbb{R}^2$.

(b) Describe the kernel of this matrix geometrically.

(c) Is the set of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfying $4y = 3x^2$ a subspace of $\mathbb{R}^2$? Justify your answer.

Solution:

(a) Let L be the line $4y = 3x$, spanned by the unit vector $\vec{u} = \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. We use the equality seen in class,

$$\text{ref}_L(\vec{v}) = 2 \cdot \text{proj}_L(\vec{v}) - \vec{v},$$

and the recipe for finding the matrix representing a linear transformation:

Step 1

$$\begin{aligned} \text{ref}_L(\vec{e_1}) &= 2 \cdot \text{proj}_L(\vec{e_1}) - \vec{e_1} \\ &= 2 \cdot (\vec{e_1} \cdot \vec{u})\vec{u} - \vec{e_1} \\ &= 2\frac{4}{5}\frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{25}\begin{bmatrix} 7 \\ 24 \end{bmatrix} \end{aligned}$$

and similarly

$$\begin{aligned} \text{ref}_L(\vec{e_2}) &= 2 \cdot \text{proj}_L(\vec{e_2}) - \vec{e_2} \\ &= 2 \cdot (\vec{e_1} \cdot \vec{u})\vec{u} - \vec{e_1} \\ &= 2\frac{3}{5}\frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{25}\begin{bmatrix} 24 \\ -7 \end{bmatrix} \end{aligned}$$

Step 2 The representing matrix has the vectors found in step 1 as column vectors, thus

$$\frac{1}{25}\begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix}$$

(b) The reflection is an invertible transformation (in fact, it is its own inverse) hence the kernel is the trivial subspace $\{\vec{0}\}$.

(c) The set of $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfying $4y = 3x^2$ is not a subspace of $\mathbb{R}^2$. To see this, notice that $\vec{v} = \begin{bmatrix} 1 \\ 3/4 \end{bmatrix}$ lies in the subspace but $2 \cdot \vec{v} = \vec{v} + \vec{v}$ does not. So the subset is not closed under vector addition nor scalar multiplication and hence doesn't satisfy two out of the three conditions for being a subspace.

4. (10 points) (a) Identify the redundant vectors in the following list:

$$\begin{bmatrix}-1\\1\\3\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\-2\\-3\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\4\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

(b) Consider the matrix $A = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}$. Compute A^{2016}.

Solution:

(a) We assemble the given vectors into a matrix (the order in which the vectors are presented must be respected) and then perform row-reduction:

$$\begin{bmatrix}-1 & 0 & -1 & 1 & 0 & 0\\1 & 0 & -2 & 1 & 2 & 0\\3 & 0 & -3 & 1 & 4 & 1\end{bmatrix} \to \begin{bmatrix}-1 & 0 & -1 & 1 & 0 & 0\\0 & 0 & -3 & 2 & 2 & 0\\3 & 0 & -3 & 1 & 4 & 1\end{bmatrix} \to \begin{bmatrix}-1 & 0 & -1 & 1 & 0 & 0\\0 & 0 & -3 & 2 & 2 & 0\\0 & 0 & -6 & 4 & 4 & 1\end{bmatrix} \to$$
$$\begin{bmatrix}-1 & 0 & -1 & 1 & 0 & 0\\0 & 0 & -3 & 2 & 2 & 0\\0 & 0 & 0 & 0 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 1 & -1 & 0 & 0\\0 & 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & 0\\0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

where we have applied the row operations $R2 = R2 + R1, R3 = R3 + 3R1$, $R3 = R3 - 2R2, R1 = -R1$, and $R2 = -\frac{1}{3}R2$ in succession.

We find pivots in the columns 1,3, and 6 hence the redundant vectors are $\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\4\end{bmatrix}$.

(b) We try to find a pattern by computing the first few powers of A.

We have $A^1 = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}$, $A^2 = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix} = \begin{bmatrix}1 & 3\\0 & 4\end{bmatrix}$, $A^3 = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 3\\0 & 4\end{bmatrix} = \begin{bmatrix}1 & 7\\0 & 8\end{bmatrix}$, $A^4 = \begin{bmatrix}1 & 7\\0 & 8\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix} = \begin{bmatrix}1 & 15\\0 & 16\end{bmatrix}$

We see that the top left entry is always 1, the bottom left entry is always 0, and the bottom right entry of A^n is 2^n. The top right entry is obtained by subtracting 1 from the bottom right entry. Thus, the formula for $A^n = \begin{bmatrix}1 & 2^n - 1\\0 & 2^n\end{bmatrix}$, so $A^{2016} = \begin{bmatrix}1 & 2^{2016} - 1\\0 & 2^{2016}\end{bmatrix}$.

We can prove our general formula by induction (this was not required to receive full credit). Let P_n denote the assertion that $A^n = \begin{bmatrix}1 & 2^n - 1\\0 & 2^n\end{bmatrix}$. Clearly, P_1 (our base case) is true. For the inductive step, assuming P_n, we see that $A^{n+1} = A^n A = \begin{bmatrix}1 & 2^n - 1\\0 & 2^n\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix} = \begin{bmatrix}1 * 1 + 0 & 1 + 2(2^n - 1)\\0 & 2 * 2^n\end{bmatrix} = \begin{bmatrix}1 & 2^{n+1} - 1\\0 & 2^{n+1}\end{bmatrix}$. Thus P_{n+1} is true, and the induction is complete.