Math 33A/1
Spring 2016
06/08/16
Time Limit: 180 Minutes

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Day \ T.A.	David	Casey	Adam
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

This exam contains 9 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, cross the box correspondiing to your discussion section, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Problem	Points	Score
1	10	10
2	10	10
3	12	12
4	12	12
5	10	6
6	8	8
7	10	9
8	8	8
Total:	80	75

1. (a) (5 points) Find all solutions to the system

$$\left|\begin{aligned} 3x_1 - 6x_2 - 12x_3 &= -9 \\ 2x_2 + 6x_3 &= 8 \\ x_1 - 4x_2 - 10x_3 &= -11 \\ 2x_1 - 2x_2 - 2x_3 &= 2 \end{aligned}\right|.$$

Let r = any const. $\begin{bmatrix} 5-2r \\ 4-3r \\ r \end{bmatrix}$ ✓

① Ex. of one sol: $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\vec{x}=\begin{bmatrix}1\\3\end{bmatrix}$

Ex. of no sol: $A=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$ $\vec{x}=\begin{bmatrix}1\\1\\5\end{bmatrix}$

(b) (5 points) Let A be an $n \times m$ matrix, $\vec{b}$ a vector in $\mathbb{R}^n$, and consider the system $A\vec{x} = \vec{b}$.

1. Suppose that $\text{rank}(A) = m$. How many solutions can the system possibly have?
2. Now suppose instead that $\text{rank}(A) = n$ and $m \neq n$. How many solutions can the system possibly have?

Don't forget to justify your answer.

in separate paper

a)

$$\begin{bmatrix} 3 & -6 & -12 & -9 \\ 0 & 2 & 6 & 8 \\ 1 & -4 & -10 & -11 \\ 2 & -2 & -2 & 2 \end{bmatrix} \downarrow \begin{bmatrix} 1 & -2 & -4 & -3 \\ 0 & 1 & 3 & 4 \\ 1 & -4 & -10 & -11 \\ 1 & -1 & -1 & 1 \end{bmatrix} \downarrow \begin{bmatrix} -1 & 2 & 4 & 3 \\ 0 & 1 & 3 & 4 \\ 1 & -4 & -10 & -11 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\downarrow \begin{bmatrix} -1 & 2 & 4 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & -2 & -6 & -8 \\ 0 & 1 & 3 & 4 \end{bmatrix} \downarrow \begin{bmatrix} -1 & 2 & 4 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -4 & -3 \\ 0 & 2 & 6 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \downarrow \begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & 3 & 4 \end{bmatrix}$$

Finding solution

$x_1 + 2x_3 = 5$
$x_2 + 3x_3 = 4$
$x_1 = 5 - 2x_3$
$x_2 = 4 - 3x_3$
Let r = constant

sol: $\begin{bmatrix} 5-2r \\ 4-3r \\ r \end{bmatrix}$

b) ① ~~The system must have exactly one solution~~

There must be m rows with leading 1s if the rank is m, meaning there are no free variables, and $n \geq m$. If $n = m$, there must be exactly one solution. If $n > m$, there could be no solution if there is inconsistency, but there cannot be infinitely many since the number of pivots equals the number of variables, so there are no free variables. ✓

② If $n < m$, there may be infinitely many solutions, as there are more variables than equations in the system. ~~If ... there could be an inconsistent system~~ ✓ There cannot be one solution due to free variables or no solution, since $n < m$.

2. (a) (7 points) Let V be the plane $2x + y - z = 0$ in $\mathbb{R}^3$. Compute the orthogonal projection onto V of the vector $\vec{v} = \begin{bmatrix}6\\0\\6\end{bmatrix}$.

   (b) (3 points) Let A be the 2×2 matrix of a rotation through 45 degrees counter-clockwise in $\mathbb{R}^2$. Compute A^4.

a) $\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}$

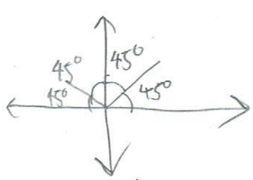
$\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \frac{2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \begin{bmatrix}1\\0\\2\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$

$u_1: \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix}, \quad u_2: \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}$ } span V

$(u_1 \cdot v)u_1 + (u_2 \cdot v)u_2$ } projection

$\frac{6}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix} + \frac{12}{\sqrt{3}}\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}0\\3\\3\end{bmatrix} + \begin{bmatrix}4\\-4\\4\end{bmatrix} = \boxed{\begin{bmatrix}4\\-1\\7\end{bmatrix}}$

b)



A^4: π, counterclockwise

$\begin{bmatrix}\cos\pi & -\sin\pi\\ \sin\pi & \cos\pi\end{bmatrix} \rightarrow \boxed{\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}}$

3. (a) (5 points) Let W be the span of the vectors

$$\vec{w}_1 = \overset{-2}{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}}, \vec{w}_2 = \overset{2}{\begin{bmatrix}2\\1\\2\\-1\end{bmatrix}}, \vec{w}_3 = \cancel{\begin{bmatrix}0\\-3\\-4\\-9\end{bmatrix}}, \vec{w}_4 = \overset{1}{\begin{bmatrix}7\\0\\0\\0\end{bmatrix}}, \vec{w}_5 = \cancel{\begin{bmatrix}9\\-2\\-2\\-10\end{bmatrix}}.$$

Find a basis for W.

(b) (1 point) Compute the dimension of $W^{\perp}$.

(c) (6 points) Compute the traces and determinants of the following matrices:

1. A represents the orthogonal projection onto the line L spanned by $\vec{v} = \begin{bmatrix}-1\\1\\-2\end{bmatrix}$ in $\mathbb{R}^3$.

2. $B = \begin{bmatrix}1&1&1&1\\2&2&2&2\\3&3&3&3\\4&4&4&4\end{bmatrix}$ 3. $C = \begin{bmatrix}1&0&1&0\\ \cancel{0}&\cancel{1}&\cancel{0}&\cancel{2}\\1&0&3&0\\4&0&0&2\end{bmatrix}$. $\to 1\cdot\det\begin{bmatrix}1&1&0\\1&3&0\\4&0&2\end{bmatrix} = 1(6)-1(2) = 4$

$$\begin{bmatrix}1&-2&0\\2&1&-3\\3&2&-4\\4&-1&-9\end{bmatrix} \to \begin{bmatrix}1&-2&0\\0&-3&-3\\0&-4&-4\\0&-9&-9\end{bmatrix}, \begin{bmatrix}1&0&-2\\0&1&1\\0&0&0\\0&0&0\end{bmatrix} \to -2\vec{w_1}+\vec{w_1}=\vec{w_3}$$

$a+2b=7$
$2a+b=0, b=-2a$
$a+2(-2a)=7$
$-3a=7, a=-\frac{7}{3}, b=\frac{14}{3}$
$3\left(-\frac{7}{3}\right)+\frac{14}{3}(2)\neq 0$
$\vec{w_4}$ not redundant.

$a+2b+7c=9$
$2a+b=-2 \to b=-2-2a$
$3a+2b=-2$
$4a-b=-10$
$3a+2(-2-2a)=-2$
$3a-4-4a=-2$
$-a=2, a=-2, b=2, c=1$
$\vec{w_5}=-2\vec{w_1}+2\vec{w_2}+\vec{w_4}$

herefore, $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}2\\1\\2\\-1\end{bmatrix}, \begin{bmatrix}7\\0\\0\\0\end{bmatrix}$ is a basis

) dim (W) = 3, thus, dim $W^{\perp}$ = 1 since the subspace is in $\mathbb{R}^4$

) ① Similar to projection onto $\vec{e}_1$ $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$, det: 0, trace: 1

② RREF: $\begin{bmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \to$ det: 0, trace = 1+2+3+4 = 10

③ Trace: 1+1+3+2 = 7
Det: 4 (work above)

$$\begin{bmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix}$$

6

4. (a) (6 points) Find the quadratic polynomial $f(t) = c_0 + c_1 t + c_2 t^2$ that best fits the points $(-2, 10)$, $(-1, 10)$, $(0, 40)$, $(1, 20)$, using least squares.

(b) (6 points) Which of the following pairs of matrices are similar? *Justify your answer.*

1

- $C_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $C_2 = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$. These are not similar since they have different traces. $tr(C_1) = 2$, while $tr(C_2) = 1$

3

- $D_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $D_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $(1-\lambda)(1-\lambda)(-1-\lambda) = 0$, $\lambda = 1, 1, -1$

  $(1-\lambda)(\lambda^2 - 1) \to \lambda = 1, 1, -1$

2

- $E_1 = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$, $E_2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

These two matrices are similar as they both are diagonalizable (D_1 is already diagonalized, while D_2 is symmetric and is thus diagonalizable by the spectral theorem.), and they both have no invariants (same eigenvalues, rank, nullity, trace, and determinant).

$\begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix}$: $\lambda^2 - 6\lambda + 9 = 0$

$\lambda = 3$

$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$:

$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $-r + s = 0$, $s = r$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

algmu of $3 \neq$ gemu(3) for E_1. Thus, E_1 is not diagonalizable and cannot be similar to E_2.

4a

$10 = c_0 - 2c_1 + 4c_2$

$10 = c_0 - c_1 + c_2$

$40 = c_0$

$20 = c_0 + c_1 + c_2$

$\to \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 10 \\ 10 \\ 40 \\ 20 \end{bmatrix}$

$A^TA \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 \\ 4 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} : \begin{bmatrix} 4 & -2 & 6 \\ -2 & 6 & -8 \\ 6 & -8 & 18 \end{bmatrix}$

$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 \\ 4 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 40 \\ 20 \end{bmatrix} = \begin{bmatrix} 80 \\ -10 \\ 70 \end{bmatrix}$

$\begin{bmatrix} 2 & -1 & 3 & 40 \\ -1 & 3 & -4 & -5 \\ 3 & -4 & 9 & 35 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 4 & 5 \\ 0 & 5 & -5 & 30 \\ 0 & 5 & -3 & 20 \end{bmatrix}$

$\begin{bmatrix} 4 & -2 & 6 & 80 \\ -2 & 6 & -8 & -10 \\ 6 & -8 & 18 & 70 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 4 & 5 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 4 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -5 \end{bmatrix}$

$\boxed{\begin{bmatrix} 28 \\ 1 \\ -5 \end{bmatrix}}$

5. (a) (5 points) Find the $\mathcal{B}$-matrix of A where

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -4 & 7 \\ 1 & -2 & 3 \end{bmatrix}, \qquad \mathcal{B} = \{\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$$

1-4

(b) (5 points) Compute the classical adjoint of $D = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ 0 & 5 & 0 \end{bmatrix}$ and use this to find D^{-1}.

a) $\begin{bmatrix} 1 & 1 & -2 \\ 1 & -4 & 7 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = 2\vec{v_2}$

$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -4 & 7 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} = \vec{V_3} - \vec{V_1}$

$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -4 & 7 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \vec{V_1}$

Thus, the B-matrix is

$$\begin{bmatrix} 0 & -1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

b) $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ 0 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 1(-20) & 0 & 1(10) \\ 2(-5) & 3(0) & 4(5) \\ 0 & 5(2) & 0 \end{bmatrix} \to \begin{bmatrix} -20 & 0 & 10 \\ -10 & 0 & 20 \\ 0 & 10 & 0 \end{bmatrix}$

signs

$\begin{bmatrix} -20 & 0 & 10 \\ 10 & 0 & -20 \\ 0 & -10 & 0 \end{bmatrix}$

-4

transpose

Classical adjoint $\boxed{\begin{bmatrix} -20 & \cancel{10} & \cancel{0} \\ 0 & 0 & \cancel{-10} \\ 10 & \cancel{-20} & \cancel{0} \end{bmatrix}}$

$\det(A) = 1(0-20) - 0(0) + 1(10) = -10$

$D^{-1} = \frac{1}{\det(A)}$ classical adj $= \frac{1}{-10}\begin{bmatrix} -20 & 10 & 0 \\ 0 & 0 & -10 \\ 10 & -20 & 0 \end{bmatrix}$

$D^{-1} = \boxed{\begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}}$

6. Let
$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}.$$

(a) (3 points) Find the eigenvalues of A. Determine their algebraic multiplicities.

(b) (3 points) Find the geometric multiplicity of each eigenvalue.

(c) (1 point) Is A diagonalizable?

(d) (1 point) Is A invertible?

a) $\begin{bmatrix} -1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & -1 & 1-\lambda \end{bmatrix} \rightarrow (1-\lambda)\det\begin{bmatrix} -1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} = 0$

$(1-\lambda)\left[(1-\lambda)(1-\lambda)(-1-\lambda) - 1(1-\lambda)\right] = 0$

$(1-\lambda)\left[(1-\lambda)\left[(1-\lambda)(-1-\lambda) - 1\right]\right] = 0$

$(1-\lambda)\left[(1-\lambda)(-1+\lambda^2-1)\right] = 0$

3

$\lambda = 1$ algmu 2 $\quad \lambda = \sqrt{2}$, $\lambda = -\sqrt{2}$ algmu: 1

3

b) The geometric multiplicities of $\sqrt{2}$ and $-\sqrt{2}$ must be one, since gemu $\leq$ algmu and cannot be 0 for the eigenvalue.

$\lambda = 1$: $\begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

kernel $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, thus the geometric multiplicity of 1 is 1.

1 c) A is not diagonalizable since the algebraic multiplicity of 1 does not equal the geometric multiplicity.

1 d) $\det(A) = \det\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = -1(1) + 1(-1) = -2$

Yes, it is invertible since the determinant $\neq 0$.

7. (a) (6 points) Let A be the matrix
$$\begin{bmatrix} -2 & -9 \\ 6 & 19 \end{bmatrix}.$$
Show that there is a matrix B such that $B^4 = A$. *Hint: You don't have to necessarily produce B explicitly. It is enough to explain why B exists.*

(b) (4 points) Find all 3×3 matrices for which both $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors with associated eigenvalue 2.

a) $\begin{bmatrix} -2 & -9 \\ 6 & 19 \end{bmatrix} \to \lambda^2 - 17\lambda + 16$, $\lambda = 1$ and $\lambda = 16$ are both distinct eigenvalues. Thus, A can be diagonalized. Let $D = S^{-1}AS$, the diagonal matrix. Then, let D_S = a diagonal matrix such that $(D_S)^4 = D$, and $D_S = S^{-1}BS$. Then, $(SBS^{-1})^4 = S^{-1}AS$, and SB^4S^{-1} is $S^{-1}AS$, showing that there is a matrix $B^4 = A$.

Why does this exist? (-1)

b) $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ $(A\vec{v}) = \lambda\vec{v}$

$a+b+c=2$, $d+e+f=2$, $g+h+i=2$

$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

$a+c=2$ $\quad b=0,\ a=2-c$

$d+f=0$ $\quad e=2,\ d=-f$

$g+i=2$ $\quad h=0,\ g=2-i$

Therefore
$$\begin{bmatrix} 2-c & 0 & c \\ -f & 2 & f \\ 2-i & 0 & i \end{bmatrix}$$

8. (a) (7 points) Let
$$q(x_1, x_2, x_3) = -x_1^2 - 4x_2^2 + 4x_2x_3 - x_3^2.$$

Diagonalize q.

(b) (1 point) Determine the definiteness of q.

7

a) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} -1-\lambda & 0 & 0 \\ 0 & -4-\lambda & 2 \\ 0 & 2 & -1-\lambda \end{bmatrix}$

$\lambda = -1$
$\lambda = 0$
$\lambda = -5$

$(-1-\lambda)\left[(-4-\lambda)(-1-\lambda) - 4\right] = 0$

$(-1-\lambda)\left(\cancel{4} + 4\lambda + \lambda + \lambda^2 - \cancel{4}\right) = 0$

$(-1-\lambda)(\lambda^2 + 5\lambda) = 0, \; \lambda(\lambda+5) = 0$

$\lambda = 0\}: \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ orthogonal.

$\lambda = -1\} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix} : \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Spectral theorem

$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{array}$ $\quad 0\vec{i} + 2\vec{j} - 1\vec{k}$

$(0, 2, -1)$

Thus, q can be diagonalized to $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$, with $S = \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 2/\sqrt{5} & 0 & -1/\sqrt{5} \end{bmatrix}$ and $S^{-1} = \begin{bmatrix} 0 & 1/\sqrt{5} & 2/\sqrt{5} \\ 1 & 0 & 0 \\ 0 & 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$

1 b) some eigenvalues are negative while another is 0. Thus, q is negative semidefinite. [Neither definite nor indefinite]