Math 33A/1
Spring 2016
06/08/16
Time Limit: 180 Minutes

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Day \ T.A.	David	Casey	Adam
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

This exam contains 9 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, cross the box correspondiing to your discussion section, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work,** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Problem	Points	Score
1	10	10
2	10	10
3	12	12
4	12	12
5	10	10
6	8	8
7	10	10
8	8	8
Total:	80	80

MW

1. (a) (5 points) Find all solutions to the system

$$\left|\begin{aligned} 3x_1 - 6x_2 - 12x_3 &= -9 \\ 2x_2 + 6x_3 &= 8 \\ x_1 - 4x_2 - 10x_3 &= -11 \\ 2x_1 - 2x_2 - 2x_3 &= 2 \end{aligned}\right|.$$

(b) (5 points) Let A be an $n \times m$ matrix, $\vec{b}$ a vector in $\mathbb{R}^n$, and consider the system $A\vec{x} = \vec{b}$.

1. Suppose that $\text{rank}(A) = m$. How many solutions can the system possibly have?
2. Now suppose instead that $\text{rank}(A) = n$ and $m \neq n$. How many solutions can the system possibly have?

Don't forget to justify your answer.

m, n, b, n

a.

$$\left[\begin{array}{ccc|c} 3 & -6 & -12 & -9 \\ 0 & 2 & 6 & 8 \\ 1 & -4 & -10 & -11 \\ 2 & -2 & -2 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -4 & -10 & -11 \\ 0 & 2 & 6 & 8 \\ 3 & -6 & -12 & -9 \\ 2 & -2 & -2 & 2 \end{array}\right] \begin{array}{c} \\ \\ -3\,I \\ -2I \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -4 & -10 & -11 \\ 0 & 2 & 6 & 8 \\ 0 & 6 & 18 & 24 \\ 0 & 6 & 18 & 24 \end{array}\right] \begin{array}{c} +2\,II \\ \div 2 \\ -3\,II \\ -3\,II \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + 2x_3 = 5$

$x_2 + 3x_3 = 4$

$x_3 = r$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5-2r \\ 4-3r \\ r \end{bmatrix}}$$ ✓

b.

1. If rank(A) = m, then the system either has no solutions or one unique solution. ✓

In this situation, $n \geq m$ since rank cannot be greater than the number of rows. ✓
A rank of m means that each column has a pivot. The rows of the bottom that do not have pivots will be 0's. If the corresponding values in $\vec{b}$ are 0, there is a unique solution and if they are not 0, there is no solution. ✓ perfect!

2. If rank(A) = n and $m \neq n$, then the system has infinitely many solutions. ✓

Since $m \neq n$ and m cannot be less than n, m must be greater than n. ✓
A rank of n means that the first n variables are dependent on the remaining variables which can be arbitrary values. Thus, there are infinitely many solutions. ✓

MW

2. (a) (7 points) Let V be the plane $2x + y - z = 0$ in $\mathbb{R}^3$. Compute the orthogonal projection onto V of the vector $\vec{v} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$.

   (b) (3 points) Let A be the 2×2 matrix of a rotation through 45 degrees counter-clockwise in $\mathbb{R}^2$. Compute A^4.

a. normal vector to plane $V = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

need two linearly independent vectors that lie in V, which is $\perp$ to $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\quad v_1 \perp v_2$ since $v_1 \cdot v_2 = 0$

$u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $\quad u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$v'' = (u_1 \cdot v)u_1 + (u_2 \cdot v)u_2 = \frac{12}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) + \frac{6}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$$

$$= 4\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}$$

b. 4 45° rotations

$45° \times 4 = 180°$

$$A^4 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta = 180$$

$$\boxed{A^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}$$

MW

3. (a) (5 points) Let W be the span of the vectors

$$\vec{w}_1 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \vec{w}_2 = \begin{bmatrix}2\\1\\2\\-1\end{bmatrix}, \vec{w}_3 = \begin{bmatrix}0\\-3\\-4\\-9\end{bmatrix}, \vec{w}_4 = \begin{bmatrix}7\\0\\0\\0\end{bmatrix}, \vec{w}_5 = \begin{bmatrix}9\\-2\\-2\\-10\end{bmatrix}.$$

Find a basis for W.

(b) (1 point) Compute the dimension of $W^{\perp}$.

(c) (6 points) Compute the traces and determinants of the following matrices:

1. A represents the orthogonal projection onto the line L spanned by $\vec{v} = \begin{bmatrix}-1\\1\\-2\end{bmatrix}$ in $\mathbb{R}^3$.

2. $B = \begin{bmatrix}1&1&1&1\\2&2&2&2\\3&3&3&3\\4&4&4&4\end{bmatrix}$ 3. $C = \begin{array}{c} \begin{array}{cccc}+&-&+&-\end{array} \\ \begin{bmatrix}1&0&1&0\\0&1&0&2\\1&0&3&0\\4&0&0&2\end{bmatrix}\end{array}$.

a. 5

$$\begin{bmatrix}1&2&0&7&9\\2&1&-3&0&-2\\3&2&-4&0&-2\\4&-1&-9&0&-10\end{bmatrix}\begin{array}{l}\\-2\mathrm{I}\\-3\mathrm{I}\\-4\mathrm{I}\end{array}$$

$$\begin{bmatrix}1&2&0&7&9\\0&-3&-3&-14&-20\\0&-4&-4&-21&-29\\0&-9&-9&-28&-46\end{bmatrix}\begin{array}{l}\\-\mathrm{III}\\ \\ \\ \end{array}$$

$$\begin{bmatrix}1&2&0&7&9\\0&1&1&7&9\\0&-4&-4&-21&-29\\0&-9&-9&-28&-46\end{bmatrix}\begin{array}{l}-2\mathrm{II}\\ \\+4\mathrm{II}\\+9\mathrm{II}\end{array}$$

$$\begin{bmatrix}1&0&-2&-7&-9\\0&1&1&7&9\\0&0&0&7&7\\0&0&0&35&35\end{bmatrix}\begin{array}{l}+\mathrm{III}\\-1\mathrm{III}\\ \div 7\\-5\mathrm{III}\end{array}$$

$$\begin{bmatrix}1&0&-2&0&-2\\0&1&1&0&2\\0&0&0&1&1\\0&0&0&0&0\end{bmatrix}$$

basis is w_1, w_2, w_4

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix} \begin{bmatrix}2\\1\\2\\-1\end{bmatrix} \begin{bmatrix}7\\0\\0\\0\end{bmatrix}$$

1 b. $\dim(W^{\perp}) = 4 - \dim(W)$
$= 4 - 3$
$= \boxed{1}$

c. 1. $v = \begin{bmatrix}-1\\1\\-2\end{bmatrix}$ $u = \frac{1}{\sqrt6}\begin{bmatrix}-1\\1\\-2\end{bmatrix}$

3

$Te_1 = (u_1 \cdot e_1)u_1 = \frac{-1}{\sqrt6}\frac{1}{\sqrt6}\begin{bmatrix}-1\\1\\-2\end{bmatrix} = \frac16\begin{bmatrix}1\\-1\\2\end{bmatrix}$

$Te_2 = (u_2 \cdot e_2)u_1 = \frac{1}{\sqrt6}\frac{1}{\sqrt6}\begin{bmatrix}-1\\1\\-2\end{bmatrix} = \frac16\begin{bmatrix}-1\\1\\-2\end{bmatrix}$

$Te_3 = (u \cdot e_3)u_1 = \frac{-2}{\sqrt6}\frac{1}{\sqrt6}\begin{bmatrix}-1\\1\\-2\end{bmatrix} = \frac13\begin{bmatrix}1\\-1\\2\end{bmatrix}$

$A = \begin{bmatrix}1/6&-1/6&1/3\\-1/6&1/6&-1/3\\1/3&-1/3&2/3\end{bmatrix}\begin{array}{l}\times6\\ \times6\\ \times6\end{array}$

$\boxed{\begin{array}{l}\text{trace}(A) = 1\\ \det(A) = 0\end{array}}$

$\det\begin{bmatrix}1&-1&2\\-1&1&-2\\2&-2&4\end{bmatrix} = 0$

2. $\boxed{\begin{array}{l}\text{trace}(B) = 10\\ \det(B) = 0\\ \text{because } B \text{ is not invertible}.\end{array}}$

1

3. $\boxed{\text{trace}(C) = 7}$

2

$1\begin{vmatrix}1&0&2\\0&3&0\\0&0&2\end{vmatrix} + 1\begin{vmatrix}0&1&2\\1&0&0\\4&0&2\end{vmatrix}$

6 -2

$6-2=4$

$\boxed{\det(C) = 4}$

MW

6

4. (a) (6 points) Find the quadratic polynomial $f(t) = c_0 + c_1 t + c_2 t^2$ that best fits the points $(-2, 10)$, $(-1, 10)$, $(0, 40)$, $(1, 20)$, using least squares.

(b) (6 points) Which of the following pairs of matrices are similar? *Justify your answer.*

- $C_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $C_2 = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$.
- $D_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $D_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
- $E_1 = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$, $E_2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ ← on back of page

a.

$c_0 - 2c_1 + 4c_2 = 10$

$c_0 - c_1 + c_2 = 10$

$c_0 = 40$

$c_0 + c_1 + c_2 = 20$

$$\begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 40 \\ 20 \end{bmatrix}$$

$A \quad x \quad b$

$A^T A x = A^T b$

$x = (A^T A)^{-1} A^T b$

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 \\ 4 & 1 & 0 & 1 \end{bmatrix} \qquad A^T b = \begin{bmatrix} 80 \\ -10 \\ 70 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4 & -2 & 6 \\ -2 & 6 & -8 \\ 6 & -8 & 18 \end{bmatrix}$$

$$A^T A: \left[\begin{array}{ccc|ccc} -2 & 6 & -8 & 0 & 1 & 0 \\ 4 & -2 & 6 & 1 & 0 & 0 \\ 6 & -8 & 18 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \times -1 \\ +2\mathrm{I} \\ +3\mathrm{I} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & -6 & 8 & 0 & -1 & 0 \\ 0 & 10 & -10 & 1 & 2 & 0 \\ 0 & 10 & -6 & 0 & 3 & 1 \end{array}\right] -\mathrm{II}$$

$$\left[\begin{array}{ccc|ccc} 2 & -6 & 8 & 0 & -1 & 0 \\ 0 & 10 & -10 & 1 & 2 & 0 \\ 0 & 0 & 4 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} \div 2 \\ \div 10 \\ \div 4 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & 0 & -1/2 & 0 \\ 0 & 1 & -1 & 1/10 & 1/5 & 0 \\ 0 & 0 & 1 & -1/4 & 1/4 & 1/4 \end{array}\right] +3\mathrm{II}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3/10 & 1/10 & 0 \\ 0 & 1 & -1 & 1/10 & 1/5 & 0 \\ 0 & 0 & 1 & -1/4 & 1/4 & 1/4 \end{array}\right] \begin{array}{l} -\mathrm{III} \\ +\mathrm{III} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 11/20 & -3/20 & -1/4 \\ 0 & 1 & 0 & -3/20 & 9/20 & 1/4 \\ 0 & 0 & 1 & -1/4 & 1/4 & 1/4 \end{array}\right]$$

$$(A^T A)^{-1} = \frac{1}{20}\begin{bmatrix} 11 & -3 & -5 \\ -3 & 9 & 5 \\ -5 & 5 & 5 \end{bmatrix} \cdot \begin{bmatrix} 80 \\ -10 \\ 70 \end{bmatrix}$$

$$(A^T A)^{-1} A^T b = \frac{1}{20}\begin{bmatrix} 560 \\ 20 \\ -100 \end{bmatrix} = \begin{bmatrix} 28 \\ 1 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 28 \\ 1 \\ -5 \end{bmatrix}$$

$$\boxed{f(t) = 28 + t - 5t^2}$$

1 b. C_1 and C_2 are not similar because their determinants are not equal.

3

D_1 and D_2 are similar because D_2 is diagonalizable and becomes D_1.

$D_1 = S^{-1} D_2 S$

$\lambda^2(\lambda - 1) - 1(\lambda - 1) = 0$

$\lambda = 1, 1$

for E ↓

$E_1 = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \quad E_2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

2

$\begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix}$

charpoly $= (2-\lambda)(4-\lambda)+1=0$

$= 8-6\lambda+\lambda^2+1=0$

$(\lambda-3)^2=0$

$\lambda = 3, 3$

$E_3 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

$\ker(E_3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

gemu$(3)=1$

$\begin{bmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix}$

charpoly $= (\lambda-3)^2=0$

$\lambda = 3, 3$

$E_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\ker(E_3) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

gemu$(3)=2$

Both E_1 and E_2 have eigenvalues of 3.

But in E_1, gemu$(3)=1$.

in E_2, gemu$(3)=2$.

$\boxed{\text{so } E_1 \text{ and } E_2 \text{ are not similar.}}$

MW

5. (a) (5 points) Find the $\mathcal{B}$-matrix of A where

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -4 & 7 \\ 1 & -2 & 3 \end{bmatrix}, \qquad \mathcal{B} = \{\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$$

(b) (5 points) Compute the classical adjoint of $D = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ 0 & 5 & 0 \end{bmatrix}$ and use this to find D^{-1}.

a. $S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

S^{-1}: $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ -\mathrm{I} \\ -\mathrm{I} \end{array}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \\ 0 & 2 & -1 & -1 & 1 & 0 \end{array}\right] \begin{array}{c} \\ \\ -2\mathrm{II} \end{array}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{array}\right] \begin{array}{c} -\mathrm{III} \\ +\mathrm{III} \\ \end{array}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{array}\right]$

$S^{-1} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix}$

$B = S^{-1} A S$

$= \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & -4 & 7 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 4 & -1 & 1 \\ 2 & -1 & 1 \end{bmatrix}$

$\boxed{B = \begin{bmatrix} 0 & -1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}$

b. $\begin{bmatrix} -20 & 0 & 10 \\ -5 & 0 & 5 \\ -3 & 2 & 3 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} -20 & -5 & -3 \\ 0 & 0 & 2 \\ 10 & 5 & 3 \end{bmatrix}$

$\downarrow$

$\boxed{\operatorname{adj}(D) = \begin{bmatrix} -20 & 5 & -3 \\ 0 & 0 & -2 \\ 10 & -5 & 3 \end{bmatrix}}$

$\det(D) = 0 + 0 + 10 - 0 - 20 - 0$

$= -10$

$D^{-1} = \frac{1}{\det(D)} \operatorname{adj}(D)$

$= \frac{-1}{10} \begin{bmatrix} -20 & 5 & -3 \\ 0 & 0 & -2 \\ 10 & -5 & 3 \end{bmatrix}$

$\boxed{D^{-1} = \begin{bmatrix} 2 & -1/2 & 3/10 \\ 0 & 0 & 1/5 \\ -1 & 1/2 & -3/10 \end{bmatrix}}$

MW

6. Let

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}.$$

(a) (3 points) Find the eigenvalues of A. Determine their algebraic multiplicities.

(b) (3 points) Find the geometric multiplicity of each eigenvalue.

(c) (1 point) Is A diagonalizable?

(d) (1 point) Is A invertible?

a. $A_3 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $A_1 = [1]$

$\begin{bmatrix} -1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix}$ $[1-\lambda]$

$\det(A_3) = (-1-\lambda)(1-\lambda)^2 - (1-\lambda)$ $\det(A_1) = 1-\lambda$

$= (-\lambda-1)(\lambda^2-2\lambda+1) - 1+\lambda$

$-\lambda^3+2\lambda^2-\lambda-\lambda^2+2\lambda-1-1+\lambda$

$-\lambda^3+\lambda^2+2\lambda-2$

$-\lambda^2(\lambda-1)+2(\lambda-1)$

$(-\lambda^2+2)(\lambda-1)$

$\det(A) = \det(A_3)\cdot\det(A_1) = (-\lambda^2+2)(\lambda-1)(1-\lambda) = 0$

$= (\lambda^2-2)(\lambda-1)^2 = 0$

$\boxed{\lambda = \sqrt{2}, -\sqrt{2}, 1, 1}$

$\boxed{\begin{array}{l} \text{algmu}(\sqrt{2}) = 1 \\ \text{algmu}(-\sqrt{2}) = 1 \\ \text{algmu}(1) = 2 \end{array}}$

b. $1 \leq \text{gemu} \leq \text{algmu}$, so

$\boxed{\begin{array}{l} \text{gemu}(\sqrt{2}) = 1 \\ \text{gemu}(-\sqrt{2}) = 1 \end{array}}$

$E_1 = \begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $+2I$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $+III$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\ker(E_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\boxed{\text{gemu}(1) = 1}$

c. A is not diagonalizable since gemu(1) $\neq$ algmu(1).

d. A is invertible because the product of eigenvalues is -2 so the determinant is -2 which $\neq 0$.

MW

7. (a) (6 points) Let A be the matrix

$$\begin{bmatrix} -2 & -9 \\ 6 & 19 \end{bmatrix}.$$

Show that there is a matrix B such that $B^4 = A$. *Hint: You don't have to necessarily produce B explicitly. It is enough to explain why B exists.*

(b) (4 points) Find all 3×3 matrices for which both $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors with associated eigenvalue 2.

a. $\begin{bmatrix} -2-\lambda & -9 \\ 6 & 19-\lambda \end{bmatrix}$

$(-2-\lambda)(19-\lambda)+54=0$

$\lambda^2-17\lambda-38+54=0$

$\lambda-17\lambda+16=0$

$(\lambda-1)(\lambda-16)=0$

$\lambda=1, 16$ ✓

Since there are 2 unique eigenvalues for the 2x2 matrix, there exists an eigenbasis S and A is diagonalizable into a diagonal matrix D. ✓

$D=S^{-1}AS$ $\quad D=\begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$

$A=SDS^{-1}$

there is a diagonal matrix C such that

$C=\sqrt[4]{D}$, $\quad C=\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ ✓

$C^4=D$

Say there is a matrix B that diagonalizes to C with the same S matrix

$C=S^{-1}BS$

$B=SCS^{-1}$

$B^4=(SCS^{-1})^4=SC^4S^{-1}$

$B^4=SDS^{-1}$

$\boxed{B^4=A}$ ✓ perfect!

b. $Av=\lambda v$

$A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}=2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad A\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}=2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ $\quad A\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}=\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

$A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$a+b+c=2$ $\quad a+c=2$
$d+e+f=2$ $\quad d+f=0$
$g+h+i=2$ $\quad g+i=2$

$\begin{cases} a+b+c=2 \\ a+c=2 \end{cases}$ $\quad \begin{cases} d+e+f=2 \\ d+f=0 \end{cases}$ $\quad \begin{cases} g+h+i=2 \\ g+i=2 \end{cases}$

$b=0$ $\quad e=2$ $\quad h=0$
$a+c=2$ $\quad d+f=0$ $\quad g+i=2$

$A=\begin{bmatrix} a & 0 & c \\ d & 2 & f \\ g & 0 & i \end{bmatrix}$

in which:

$a+c=2$

$d+f=0$

$g+i=2$ ✓

MW

8

8. (a) (7 points) Let
$$q(x_1, x_2, x_3) = -x_1^2 - 4x_2^2 + 4x_2x_3 - x_3^2.$$
Diagonalize q.

(b) (1 point) Determine the definiteness of q.

a. $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & 2 & -1 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} -1-\lambda & 0 & 0 \\ 0 & -4-\lambda & 2 \\ 0 & 2 & -1-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (-4-\lambda)(-1-\lambda)^2 - 4(-1-\lambda) = 0$

$(-1-\lambda)\left[(-4-\lambda)(-1-\lambda) - 4\right] = 0$

$(-1-\lambda)\left[4 + 5\lambda + \lambda^2 - 4\right] = 0$

$(-1-\lambda)(\lambda^2 + 5\lambda) = 0$

$(-1-\lambda)\lambda(\lambda+5) = 0$

$\lambda = -1, 0, -5$

$E_{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

$\ker(E_{-1}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$E_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & 2 & -1 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -4 & 2 \end{bmatrix} \begin{matrix} \times -1 \\ \\ +2 \mathrm{I} \end{matrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\ker(E_0) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$E_{-5} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$

$\ker(E_{-5}) = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$

$q = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2$

$\lambda = -1$ $\quad$ $\lambda = 0$ $\quad$ $\lambda = -5$

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ $\quad v_3 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$

all 3 vectors orthogonal

$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad u_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ $\quad u_3 = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$

$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}$ $\quad S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} \\ 0 & 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$

$D = S^{-1}AS$ $\quad S^T = S$

$A = SDS^{-1}$

$A = SDS^T$

$q(\vec{x}) = \vec{x}^T A \vec{x}$

$q(\vec{x}) = x^T S D S^T \vec{x}$ $\quad S^T\vec{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} \\ 0 & 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ \frac{1}{\sqrt{5}}x_2 + \frac{2}{\sqrt{5}}x_3 \\ \frac{2}{\sqrt{5}}x_2 - \frac{1}{\sqrt{5}}x_3 \end{bmatrix} \begin{matrix} c \\ c \\ c \end{matrix}$

$$\boxed{q(\vec{x}) = -1(x_1)^2 - 5\left(\frac{2}{\sqrt{5}}x_2 - \frac{1}{\sqrt{5}}x_3\right)^2}$$

b. q is negative semidefinite

since eigenvalues are nonpositive.