33A-2 Second Midterm

November 20th, 2020

Read before starting the exam. Please use separate sheets to solve the problems and show your work. On the top of the first sheet, please copy the following honor statement: *I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.* Then please sign below the statement. If it is not signed, the evaluation should be given a failing grade. On the top right corner of each sheet, please write down your name with UID.

The exam is designed to be finished within 50 minutes. In the exam, please show all your work. Unjustified answers are not correct. Make clear what your final answer is.

After finishing the exam, you should upload solutions to Gradescope between November 20th 10:00 am and November 21st 9:59 am (PST). You will lose 1 point per minute of delay for submission. It is your responsibility to upload solutions correctly. When uploading, you should choose the right page for each problem; otherwise you will automatically get the zeros.

Problem 1.1 (5 points)**.** Let

$$A = \begin{bmatrix} 3 & 0 & 0 & -2 \\ -3 & 1 & -2 & -3 \\ -2 & 0 & 0 & 2 \\ 0 & 1 & -2 & -5 \end{bmatrix}.$$

1. Perform the Gauss-Jordan elimination method to compute the reduced row echelon form of A.

2. Find a basis of the kernel $\ker(A)$. Compute $\dim\ker(A)$.

3. Find a basis of the image $\operatorname{im}(A)$. Compute $\dim\operatorname{im}(A)$.

In the next problem, we denote by $\operatorname{proj}_V : \mathbb{R}^n \to \mathbb{R}^n$ the orthogonal projection onto a linear subspace V of $\mathbb{R}^n$. Suppose that $\vec{u_1}, \cdots, \vec{u_r}$ form an orthonormal basis of V. Then we have a formula

$$\operatorname{proj}_V \vec{v} = (\vec{u_1} \cdot \vec{v})\vec{u_1} + \cdots + (\vec{u_r} \cdot \vec{v})\vec{u_r}$$

for a vector $\vec{v}$ in $\mathbb{R}^n$.

Problem 1.2 (5 points)**.** Let V be a linear subspace of $\mathbb{R}^4$ spanned by

$$\vec{v_1} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ -1 \end{bmatrix}, \ \vec{v_2} = \begin{bmatrix} -4 \\ 3 \\ 4 \\ 2 \end{bmatrix}, \ \vec{v_3} = \begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$$

1. Perform the Gram-Schmidt process to find an orthonormal basis $\mathfrak{B} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ of V.
2. Let $\vec{v}$ be a vector in $\mathbb{R}^4$ defined as

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

   Compute the dot products $\vec{u_1} \cdot \vec{v}$, $\vec{u_2} \cdot \vec{v}$, $\vec{u_3} \cdot \vec{v}$.

3. Compute the $\mathfrak{B}$-coordinate vector $[\text{proj}_V \vec{v}]_{\mathfrak{B}}$ and its length $||\,[\text{proj}_V \vec{v}]_{\mathfrak{B}}\,||$.
4. Compute $\text{proj}_V \vec{v}$ and its length $||\text{proj}_V \vec{v}||$.

Problem 1.3 (5 points)**.** Let

$$A_t = \begin{bmatrix} 1-t & \frac{1}{2} & \frac{1}{4} \\ 2 & 1-t & \frac{1}{2} \\ 4 & 2 & 1-t \end{bmatrix}.$$

1. Compute the determinant $\det(A_t)$.
2. Determine all t such that A_t is invertible.
3. Determine all possible values of $\dim \ker(A_t)$. Justify your answer.

Problem 1.4 (5 points)**.** The goal of this problem is to show that any linear subspace of $\mathbb{R}^n$ is the kernel of some linear transformation.

First we consider a special case. Let V be a 2-dimensional linear subspace of $\mathbb{R}^3$ with a basis $\{\vec{v_1}, \vec{v_2}\}$. Let A be a 3×2 matrix whose column vectors are $\vec{v_1}, \vec{v_2}$.

1. Determine the reduced row echelon form $\text{rref}(A)$ of A and describe the column vectors $\vec{u_1}, \vec{u_2}$ of $\text{rref}(A)$.
2. Give an example of a linear transformation $T_0 \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $\ker(T_0)$ is spanned by $\vec{u_1}, \vec{u_2}$.
3. Explain how to construct a linear transformation $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $\ker(T) = V$ (Hint: there is an elementary 3×3 matrix E such that $\text{rref}(A) = EA$; let $T_1 \colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation represented by E and use T_0 and T_1 to construct a linear transformation with the desired property; note that T_1 is invertible since any elementary matrix is invertible).

Finally we consider a general case. Let V be an r-dimensional linear subspace of $\mathbb{R}^n$ with a basis $\{\vec{v_1}, \cdots, \vec{v_r}\}$. Let A be an $n \times r$ matrix whose column vectors are $\vec{v_1}, \cdots, \vec{v_r}$.

4. Follow the steps 1, 2, 3 as above and explain how to construct a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\ker(T) = V$.