33A-2 First Midterm (with solutions)

October 23rd, 2020

Read before starting the exam. Please use separate sheets to solve the problems and show your work. On the top of the first sheet, please copy the following honor statement: *I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.* Then please sign below the statement. If it is not signed, the evaluation should be given a failing grade. On the top right corner of each sheet, please write down your name with UID.

The exam is designed to be finished within 50 minutes. In the exam, please show all your work. Unjustified answers are not correct. Make clear what your final answer is.

After finishing the exam, you should upload solutions to Gradescope between October 23rd 10:00 am and October 24th 9:59 am (PDT). You will lose 1 point per minute of delay for submission. It is your responsibility to upload solutions correctly. When uploading, you should choose the right page for each problem; otherwise you will automatically get the zeros.

Problem 1.1. Consider the linear system given by

$$\begin{bmatrix} 0 & -4 & 2 \\ -5 & -1 & 3 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -12 \\ -13 \\ 15 \end{bmatrix}.$$

1. Determine the reduced row echelon form of the augmented matrix

$$\left[\begin{array}{ccc|c} 0 & -4 & 2 & -12 \\ -5 & -1 & 3 & -13 \\ 3 & 3 & -3 & 15 \end{array}\right]$$

by performing the Gauss-Jordan elimination method. Determine whether the linear system is consistent or inconsistent.

2. Answer the rank of the coefficient matrix

$$\begin{bmatrix} 0 & -4 & 2 \\ -5 & -1 & 3 \\ 3 & 3 & -3 \end{bmatrix}.$$

3. Determine whether the linear system has a unique solution, infinitely many solutions, or no solution. Write the solution(s) in the vector form, if any.

Solution. 1. We perform the Gauss-Jordan elimination method:

$$\left[\begin{array}{ccc|c} 0 & -4 & 2 & -12 \\ -5 & -1 & 3 & -13 \\ 3 & 3 & -3 & 15 \end{array}\right] \div(-4)$$

$$\left[\begin{array}{ccc|c} 0 & 1 & -\frac{1}{2} & 3 \\ -5 & -1 & 3 & -13 \\ 3 & 3 & -3 & 15 \end{array}\right] \begin{array}{c} \\ (\mathrm{I}) \\ -3(\mathrm{I}) \end{array}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & -\frac{1}{2} & 3 \\ -5 & 0 & \frac{5}{2} & -10 \\ 3 & 0 & -\frac{3}{2} & 6 \end{array}\right] \div(-5)$$

$$\left[\begin{array}{ccc|c} 0 & 1 & -\frac{1}{2} & 3 \\ 1 & 0 & -\frac{1}{2} & 2 \\ 3 & 0 & -\frac{3}{2} & 6 \end{array}\right] \begin{array}{c} \\ \\ -3(\mathrm{II}) \end{array}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & -\frac{1}{2} & 3 \\ 1 & 0 & -\frac{1}{2} & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{interchange (I) with (II)}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 2 \\ 0 & 1 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The reduced row echelon form is:

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 2 \\ 0 & 1 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Moreover the system is consistent.

2. The rank of the coefficient matrix is the number of the leading 1's in the reduced row echelon form, which is 2.

3. The linear system is consistent and the rank of the coefficient matrix is $2 < 3$, so it has infinitely many solutions. We can express the solutions in the vector form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix},$$

where t is any scalar.

□

Problem 1.2. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation given by

$$T(\vec{x}) = \begin{bmatrix} 1 & -1 & 0 \\ 3 & -4 & -2 \\ 2 & -4 & -3 \end{bmatrix} \vec{x}.$$

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1. Compute the inverse matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ 3 & -4 & -2 \\ 2 & -4 & -3 \end{bmatrix}^{-1}$$

   by performing the Gauss-Jordan elimination method.

2. Find vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ in $\mathbb{R}^3$ such that

$$T(\vec{v_1}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, T(\vec{v_2}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, T(\vec{v_3}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Solution. 1. We consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 3 & -4 & -2 & 0 & 1 & 0 \\ 2 & -4 & -3 & 0 & 0 & 1 \end{array}\right]$$

and perform the Gauss-Jordan elimination method:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 3 & -4 & -2 & 0 & 1 & 0 \\ 2 & -4 & -3 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -3(\mathrm{I}) \\ -2(\mathrm{I}) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & -3 & 1 & 0 \\ 0 & -2 & -3 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \div(-1) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 \\ 0 & -2 & -3 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} +(\mathrm{II}) \\ \\ +2(\mathrm{II}) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 4 & -1 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 4 & -2 & 1 \end{array}\right] \begin{array}{l} -2(\mathrm{III}) \\ -2(\mathrm{III}) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 3 & -2 \\ 0 & 1 & 0 & -5 & 3 & -2 \\ 0 & 0 & 1 & 4 & -2 & 1 \end{array}\right].$$

We therefore obtain the inverse matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ 3 & -4 & -2 \\ 2 & -4 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} -4 & 3 & -2 \\ -5 & 3 & -2 \\ 4 & -2 & 1 \end{bmatrix}.$$

2. The condition on $\vec{v_1}, \vec{v_2}, \vec{v_3}$ can be rephrased as the identity

$$\begin{bmatrix} 1 & -1 & 0 \\ 3 & -4 & -2 \\ 2 & -4 & -3 \end{bmatrix} \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solving the identity, we obtain

$$\begin{aligned} \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix} &= \begin{bmatrix} 1 & -1 & 0 \\ 3 & -4 & -2 \\ 2 & -4 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 3 & -4 & -2 \\ 2 & -4 & -3 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} -4 & 3 & -2 \\ -5 & 3 & -2 \\ 4 & -2 & 1 \end{bmatrix}. \end{aligned}$$

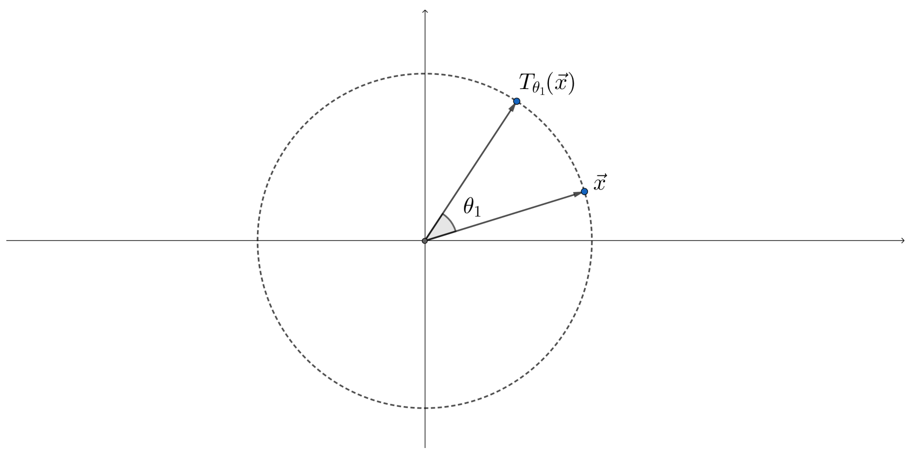
Therefore we conclude

$$\vec{v_1} = \begin{bmatrix} -4 \\ -5 \\ 4 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}, \vec{v_1} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}.$$

$\square$

The next two problems will be on rotations and reflections in $\mathbb{R}^2$. We summarize here the notation and basic properties:

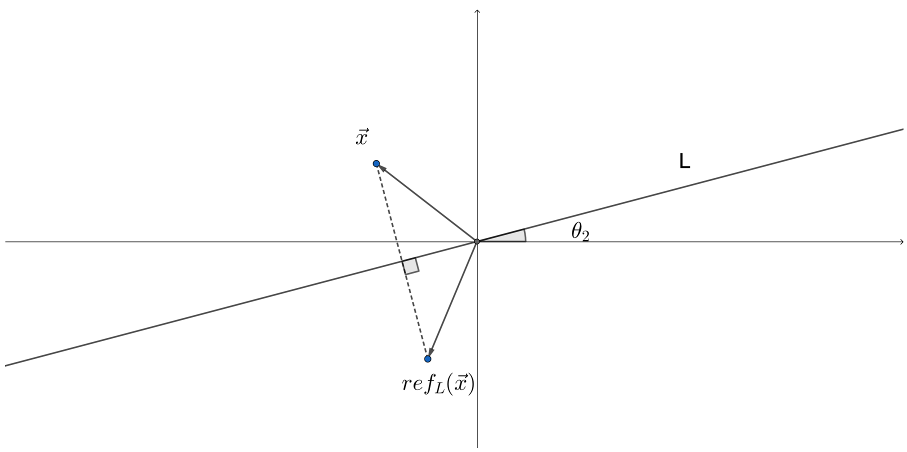
- We denote by $T_{\theta_1} : \mathbb{R}^2 \to \mathbb{R}^2$ the counter-clockwise rotation through θ_1.



The linear transformation T_{θ_1} is represented by the matrix

$$\begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}.$$

- We denote by $\mathrm{ref}_L \colon \mathbb{R}^2 \to \mathbb{R}^2$ the reflection about a line L in $\mathbb{R}^2$ through the origin. Suppose that L is obtained by rotating the x-axis through θ_2 counter-clockwise.



Then the linear transformation ref_L is represented by the matrix

$$\begin{bmatrix} \cos(2\theta_2) & \sin(2\theta_2) \\ \sin(2\theta_2) & -\cos(2\theta_2) \end{bmatrix}.$$

Problem 1.3. Deduce the triple angle formula for cosine and sine functions by the following steps.

1. Express $T_{3\theta}$ as a composition of linear transformations, using T_θ.
2. Express the matrix representing $T_{3\theta}$ as a matrix product, using the matrix representing T_θ. Compute the matrix product.
3. Based on the result of 2, express $\cos(3\theta)$ in terms of $\cos(\theta)$. Similarly, express $\sin(3\theta)$ in terms of $\sin(\theta)$.

Solution. 1. $T_{3\theta} = T_\theta \circ T_\theta \circ T_\theta$.

2. The identity $T_{3\theta} = T_\theta \circ T_\theta \circ T_\theta$ corresponds to the identity

$$\begin{bmatrix} \cos(3\theta) & -\sin(3\theta) \\ \sin(3\theta) & \cos(3\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}^3. \tag{1}$$

We compute

$$\begin{aligned}
\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}^3 &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos^2(\theta) - \sin^2(\theta) & -2\sin(\theta)\cos(\theta) \\ 2\sin(\theta)\cos(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{bmatrix}\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos^3(\theta) - 3\sin^2(\theta)\cos(\theta) & \sin^3(\theta) - 3\cos^2(\theta)\sin(\theta) \\ -\sin^3(\theta) + 3\cos^2(\theta)\sin(\theta) & \cos^3(\theta) - 3\sin^2(\theta)\cos(\theta) \end{bmatrix}. \qquad (2)
\end{aligned}$$

3. From (1) and (2), we get the identity

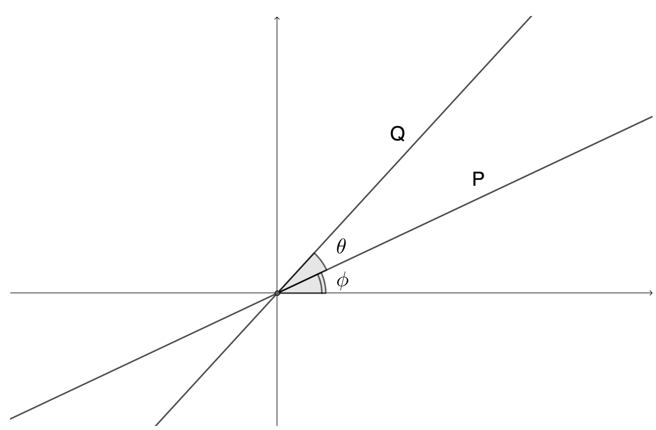
$$\cos(3\theta) = \cos^3(\theta) - 3\sin^2(\theta)\cos(\theta),\ \sin(3\theta) = -\sin^3(\theta) + 3\cos^2(\theta)\sin(\theta).$$

Using $\cos^2(\theta) + \sin^2(\theta) = 1$, one can simplify

$$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta),\ \sin(3\theta) = -4\sin^3(\theta) + 3\sin(\theta).$$

□

Problem 1.4. Let P and Q be lines in $\mathbb{R}^2$ through the origin. Suppose that Q is obtained by rotating P through θ counter-clockwise.



Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $T(\vec{x}) = \mathrm{ref}_Q(\mathrm{ref}_P(\vec{x}))$, that is, $T = \mathrm{ref}_Q \circ \mathrm{ref}_P$.

1. Suppose that P is obtained by rotating the x-axis through ϕ counter-clockwise. Answer the matrices representing ref_P and ref_Q using θ and ϕ.

2. Express the matrix representing T as a matrix product of matrices representing ref_P and ref_Q.

3. Compute the matrix product obtained in 2. Simplify the resulting matrix by applying the difference formulas:

$$\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta),$$
$$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta).$$

4. Explain the linear transformation T using the word "rotation".

5. Let $L\colon \mathbb{R}^2\to\mathbb{R}^2$ be a linear transformation given by $L(\vec{x})=\mathrm{ref}_P(\mathrm{ref}_Q(\vec{x}))$, that is, $L=\mathrm{ref}_P\circ\mathrm{ref}_Q$. Explain the linear transformation L using the word "rotation".

Solution. 1. Notice that Q is obtained by rotating the x-axis through $\theta+\phi$ counter-clockwise. The matrices representing ref_P and ref_Q are respectively

$$\begin{bmatrix}\cos(2\phi) & \sin(2\phi)\\ \sin(2\phi) & -\cos(2\phi)\end{bmatrix},\ \begin{bmatrix}\cos(2(\theta+\phi)) & \sin(2(\theta+\phi))\\ \sin(2(\theta+\phi)) & -\cos(2(\theta+\phi))\end{bmatrix}.$$

2. Since $T=\mathrm{ref}_Q\circ\mathrm{ref}_P$, the matrix representing T is equal to the matrix product

$$\begin{bmatrix}\cos(2(\theta+\phi)) & \sin(2(\theta+\phi))\\ \sin(2(\theta+\phi)) & -\cos(2(\theta+\phi))\end{bmatrix}\begin{bmatrix}\cos(2\phi) & \sin(2\phi)\\ \sin(2\phi) & -\cos(2\phi)\end{bmatrix}.$$

3. We compute the matrix product

$$\begin{aligned}&\begin{bmatrix}\cos(2(\theta+\phi)) & \sin(2(\theta+\phi))\\ \sin(2(\theta+\phi)) & -\cos(2(\theta+\phi))\end{bmatrix}\begin{bmatrix}\cos(2\phi) & \sin(2\phi)\\ \sin(2\phi) & -\cos(2\phi)\end{bmatrix}\\ =&\begin{bmatrix}\cos(2(\theta+\phi))\cos(2\phi)+\sin(2(\theta+\phi))\sin(2\phi) & \cos(2(\theta+\phi))\sin(2\phi)-\sin(2(\theta+\phi))\cos(2\phi)\\ \sin(2(\theta+\phi))\cos(2\phi)-\cos(2(\theta+\phi))\sin(2\phi) & \cos(2(\theta+\phi))\cos(2\phi)+\sin(2(\theta+\phi))\sin(2\phi)\end{bmatrix},\end{aligned}$$

which can be simplified using the difference formula as

$$\begin{bmatrix}\cos(2(\theta+\phi)-2\phi) & -\sin(2(\theta+\phi)-2\phi)\\ \sin(2(\theta+\phi)-2\phi) & \cos(2(\theta+\phi)-2\phi)\end{bmatrix}=\begin{bmatrix}\cos(2\theta) & -\sin(2\theta)\\ \sin(2\theta) & \cos(2\theta)\end{bmatrix}.$$

4. T is the counter-clockwise rotation through 2θ.

5. Notice that

$$T\circ L=\mathrm{ref}_Q\circ\mathrm{ref}_P\circ\mathrm{ref}_P\circ\mathrm{ref}_Q=\mathrm{ref}_Q\circ\mathrm{ref}_Q=\mathrm{id},$$

where we use $\mathrm{ref}_P\circ\mathrm{ref}_P=\mathrm{ref}_Q\circ\mathrm{ref}_Q=\mathrm{id}$ ($\mathrm{id}\colon\mathbb{R}^2\to\mathbb{R}^2$ is the linear transformation associating with each $\vec{x}$ in $\mathbb{R}^2$ $\vec{x}$ itself). It follows that L is the inverse T^{-1}, thus the clockwise rotation through 2θ. $\square$