20F-MATH33A-2 Final

NICHOLAS NHIEN

TOTAL POINTS

30 / 30

QUESTION 1

1 Problem 1.1 5 / 5

part 1

- **0.25 pts** One incorrect rank
- **0.5 pts** Two incorrect ranks
- **0.5 pts** No justification
- **0.25 pts** Unclear justification

part 2

- **0.5 pts** One incorrect dimension
- **1 pts** Two incorrect dimensions
- **1 pts** No justification.
- **0.5 pts** Unclear justification
- **0.25 pts** Minor errors

part 3

- **0.25 pts** Incorrect determinant
- **0.5 pts** No work on determinant
- **0.5 pts** Incorrect description of V_3
- **0.75 pts** No justification

part 4

- **0.5 pts** Incorrect description on how V_1 and V_2 intersect
- **0.25 pts** A minor error

✓ - **0 pts Correct**

QUESTION 2

2 Problem 1.2 5 / 5

✓ - **0 pts Correct**

- **1 pts** part 4: wrong
- **0.25 pts** minor mistake in R
- **0.5 pts** Part 4: computational error
- **0.5 pts** part 2: computational error (u_3)
- **0.5 pts** part 2: wrong R
- **2.5 pts** part 2: wrong
- **1 pts** part 3: wrong
- **2 pts** part 2: wrong set up and wrong u_2, u_3
- **1 pts** part 2: computational error
- **0.5 pts** Part 2 wrong, but did the correct thing in part 3
- **1.5 pts** part 2: they are orthogonal, but not normal
- **1 pts** part 3: missing
- **1 pts** part 4: missing
- **0.5 pts** part 4: half missing

QUESTION 3

3 Problem 1.3 5 / 5

part 1

- **1 pts** Incorrect inverse
- **0.5 pts** Minor errors

part 2

- **1 pts** Incorrect determinant
- **0.5 pts** Minor errors

part 3

- **1 pts** Incorrect minors
- **1 pts** Incorrect adjoint
- **0.5 pts** Multiple minor errors
- **0.25 pts** One minor error

part 4

- **0.5 pts** Incorrect inverse
- **0.5 pts** No justification (no indication of the formula $$A^{-1}=\frac{1}{\det(A)} \text{adj}(A)$$).
- **0.25 pts** minor errors

✓ - **0 pts Correct**

QUESTION 4

4 Problem 1.4 5 / 5

✓ - **0 pts Correct**

- **1 pts** Characteristic polynomial

- **0.5 pts** Eigenvalues
- **0.5 pts** alg multiplicities
- **1 pts** geometric mult.
- **1 pts** diagonalizable?
- **1 pts** fifth power
- **0.5 pts** Partial credit on above.

QUESTION 5

5 Problem 1.5 **5 / 5**

part 1

- **0.25 pts** Unclear justification, but correct example
- **0.5 pts** Incorrect example, but good attempt
- **1.25 pts** Incorrect

part 2

- **0.25 pts** Unclear justification, but correct example
- **0.5 pts** Incorrect example, but good attempt
- **1.25 pts** Incorrect

part 3

- **0.25 pts** Unclear justification, but correct example.
- **0.5 pts** Incorrect example, but good attempt
- **1.25 pts** Incorrect

part 4

- **0.25 pts** Unclear justification, but correct example
- **0.5 pts** Incorrect example, but good attempt
- **1.25 pts** Incorrect

✓ **- 0 pts Correct**

QUESTION 6

6 Problem 1.6 **5 / 5**

part 1

- **0.5 pts** incorrect or incomplete list of eigenvalues
- **1 pts** incorrect characteristic polynomial

part 2

- **0.5 pts** one incorrect basis
- **1 pts** two incorrect bases
- **1.5 pts** three incorrect bases
- **1.5 pts** missing

part 3

- **0.25 pts** does not recognise non-orthonormality or non-basis (if earlier error was made)
- **0.5 pts** incorrect orthonormal basis
- **0.5 pts** incorrect orthogonal diagonalisation
- **1 pts** missing

part 4

- **0.25 pts** indicated general calculation but only computed (up to taking the n-th power of the diagonal) for a specific value of n
- **0.25 pts** did not write down an expression for the n-th power of the relevant diagonal matrix
- **0.25 pts** finds the correct pattern but does not fully justify it or obtain a closed-form result
- **0.5 pts** did not find a correct non-trivial expression for A^n
- **0.5 pts** insufficient or incorrect justification
- **0.75 pts** only computed for a specific value of n without correct general calculation
- **1 pts** missing

✓ **- 0 pts Correct**

- **2 pts** Modifies the original matrix to a non-similar matrix in a way that leads to non-existence of an orthogonal diagonalisation

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I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation

Problems are done like [diagram of two columns with arrows], i.e. if in the middle of problem at end of left column by end of page, cont'd at top of right column

1.1

1)

$A_1 =$
-2 -1 3 2
0 2 0 3
-2 0 3 0.5
0 2 0 3

$\boxed{\text{rank}(A_1) = 2}$

$A_2 =$
-3 -3 3 0
0 -2 -1 1
-3 0 5
0 -2 -1 1

$\boxed{\text{rank}(A_2) = 2}$

2) A_1, A_2 are 2x4 matrices

By rank-nullity thm,

$\dim \ker(A) + \dim \text{im}(A) = 4$

$= \text{rank}(A)$

$= 2$

$\dim \ker(A) = 4 - 2$

$= 2$

(Since rank of both matrices = 2, calculation holds for both)

3)

$$A_3 = \begin{bmatrix} -2 & -1 & 3 & 2 \\ 0 & 2 & 0 & 3 \\ -3 & -3 & 3 & 0 \\ 0 & -2 & -1 & 1 \end{bmatrix} \begin{matrix} i \\ ii \\ iii \\ iv \end{matrix}$$

iii $- \frac{3}{2}$ i ↓

$\frac{6}{2} - \frac{9}{2} = \frac{3}{2}$

-2 -1 3 2
0 2 0 3
0 $-\frac{3}{2}$ $-\frac{3}{2}$ -3
0 -2 -1 1

$$-2\left(\det\begin{pmatrix} 2 & 0 & 3 \\ -3/2 & -3/2 & -3 \\ -2 & -1 & 1 \end{pmatrix}\right) = -2\left(2\left(\det\begin{pmatrix} -3/2 & -3 \\ -1 & 1 \end{pmatrix}\right) + 3\left(\det\begin{pmatrix} -3/2 & -3/2 \\ -2 & -1 \end{pmatrix}\right)\right)$$

$-\frac{3}{2} - \frac{6}{2}$ $\quad$ $\frac{3}{2} - \frac{6}{2}$

$$= -2\left(2\left(-\frac{9}{2}\right) + 3\left(-\frac{3}{2}\right)\right)$$

$$= -2\left(-9 - \frac{9}{2}\right) = -2\left(-\frac{18}{2} - \frac{9}{2}\right) = -2\left(-\frac{27}{2}\right) = \boxed{27}$$

(3) cont'd above

cont'd from lower-left corner

3)

$2x - \frac{3}{2} = 0$
$2x = \frac{3}{2}$
$x = \frac{3}{4}$

-2 -1 3 2
0 2 0 3 $\quad$ i $+ \frac{1}{2}$(ii)
0 $-\frac{3}{2}$ $-\frac{3}{2}$ -3 $\quad$ iii $+ \frac{3}{4}$(ii)
0 -2 -1 1 $\quad$ iv $+ 1$(ii)

-2 0 3 3.5
0 2 0 3 $\quad$ $+2$(iii)
0 0 -1.5 -.75
0 0 -1 4

$-\frac{12}{4} + \frac{9}{4} = -\frac{3}{4}$

$-\frac{2}{3}$(iii) $\quad$ $-\frac{3}{2}(x) = 1$, $x = -\frac{2}{3}$

$\frac{14}{4} - \frac{6}{4} = \frac{8}{4} = 2$

-2 0 0 2
0 2 0 3
0 0 -1.5 -.75
0 0 0 3.5

$-\frac{3}{4} - \frac{2}{3} = \frac{6}{12} = \frac{1}{2}$

-2
2 0
-1.5
3.5

$\boxed{\ker(A_3) = \{\vec{0}\}}$

4) $\boxed{\text{They intersect at a point.}}$

1 Problem 1.1 **5 / 5**

part 1

- **0.25 pts** One incorrect rank
- **0.5 pts** Two incorrect ranks
- **0.5 pts** No justification
- **0.25 pts** Unclear justification

part 2

- **0.5 pts** One incorrect dimension
- **1 pts** Two incorrect dimensions
- **1 pts** No justification.
- **0.5 pts** Unclear justification
- **0.25 pts** Minor errors

part 3

- **0.25 pts** Incorrect determinant
- **0.5 pts** No work on determinant
- **0.5 pts** Incorrect description of $$V_3$$
- **0.75 pts** No justification

part 4

- **0.5 pts** Incorrect description on how $$V_1$$ and $$V_2$$ intersect
- **0.25 pts** A minor error

✓ **- 0 pts** Correct

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1.2

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 7 \\ 0 & 1 & -2 \end{bmatrix}$$

1)
$$\begin{matrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & -5 & 5 \\ 0 & 1 & -2 \end{matrix}$$
$$\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & -2 \end{matrix}$$
$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}$$

$$\boxed{\text{rank}(A) = 3}$$

2) $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 1 \\ 0 \\ 7 \\ -2 \end{bmatrix}$

$$|v_1| = \sqrt{1+4} = \sqrt{5}$$

$$\boxed{u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}}$$

$$y_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \left(\frac{1}{\sqrt{5}} \left(\begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \right) \right) \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$
$$\left(\frac{1}{\sqrt{5}} (3+2) \right) \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

$$y_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{5}{5} \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

$$|y_2| = \sqrt{4+1+1+1} = \sqrt{7}$$

(2) cont'd above

2) cont'd from below

$$|y_2| = \sqrt{7} \qquad y_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\boxed{u_2 = \frac{1}{\sqrt{7}} \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}}$$

$$y_3 = \begin{pmatrix} 1 \\ 0 \\ 7 \\ -2 \end{pmatrix} - \left(\frac{1}{\sqrt{7}} \left(\begin{pmatrix} 1 \\ 0 \\ 7 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right) \right) \frac{1}{\sqrt{7}} \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix} - \left(\frac{1}{\sqrt{5}} \left(\begin{pmatrix} 1 \\ 0 \\ 7 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \right) \right) \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

$$= v_3 - \left(\frac{1}{\sqrt{7}} \left(2 + (-7) + (-2) \right) \right) \frac{1}{\sqrt{7}} y_2 - \left(\frac{1}{\sqrt{5}} (1+14) \right) \frac{1}{\sqrt{5}} y_1 \quad (y_1 = v_1)$$

$$= v_3 - (-1) y_2 - 3 y_1$$

$$= v_3 + y_2 - 3 y_1$$

$$= \begin{pmatrix} 1 \\ 0 \\ 7 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 1 \\ 6 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 6 \\ 0 \end{pmatrix}$$

$$y_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \qquad |y_3| = \sqrt{2}$$

$$\boxed{u_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}}$$

$$\boxed{\text{Basis} = \{u_1, u_2, u_3\}}$$

1.2 cont'd next sheet

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1.2 (cont'd)

3)

$$\boxed{Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{7}} & 0 \\ 0 & \frac{1}{\sqrt{7}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{7}} & 0 \\ 0 & \frac{1}{\sqrt{7}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \quad R = \begin{bmatrix} \sqrt{5} & \sqrt{5} & 3\sqrt{5} \\ 0 & \sqrt{7} & -\sqrt{7} \\ 0 & 0 & \sqrt{2} \end{bmatrix}}$$

$$\frac{15}{\sqrt{5}} = \frac{15\sqrt{5}}{5}$$

for $A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 7 \\ 0 & 1 & -2 \end{bmatrix}$

4) $A^T = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 3 & 1 & 1 & 1 \\ 1 & 0 & 7 & -2 \end{bmatrix}$ (By thm 5.4.5, $A^TA\vec{x} = A^T\vec{b}$ give least square solutions)

$$A^TA = \underset{3\times 4}{\begin{bmatrix} 1 & 0 & 2 & 0 \\ 3 & 1 & 1 & 1 \\ 1 & 0 & 7 & -2 \end{bmatrix}} \overset{4\times 3}{\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 7 \\ 0 & 1 & -2 \end{bmatrix}} = \begin{bmatrix} 5 & 5 & 15 \\ 5 & 12 & 8 \\ 15 & 8 & 54 \end{bmatrix}$$

$$A^T\vec{b} = \underset{3\times 4}{\begin{bmatrix} 1 & 0 & 2 & 0 \\ 3 & 1 & 1 & 1 \\ 1 & 0 & 7 & -2 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}} = \begin{bmatrix} 5 \\ 7 \\ 13 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 5 & 5 & 15 & 5 \\ 5 & 12 & 8 & 7 \\ 15 & 8 & 54 & 13 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 5 & 12 & 8 & 7 \\ 15 & 8 & 54 & 13 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 7 & -7 & 2 \\ 0 & -7 & 9 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 7 & 7 & 21 & 7 \\ 0 & 7 & -7 & 2 \\ 0 & -7 & 9 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 7 & 0 & 28 & 5 \\ 0 & 7 & -7 & 2 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 7 & 0 & 0 & 5 \\ 0 & 7 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5/7 \\ 0 & 1 & 0 & 2/7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\boxed{x^* = \begin{pmatrix} 5/7 \\ 2/7 \\ 0 \end{pmatrix}}$$

2 Problem 1.2 **5 / 5**

- ✓ **- 0 pts** Correct
- **- 1 pts** part 4: wrong
- **- 0.25 pts** minor mistake in R
- **- 0.5 pts** Part 4: computational error
- **- 0.5 pts** part 2: computational error (u_3)
- **- 0.5 pts** part 2: wrong R
- **- 2.5 pts** part 2: wrong
- **- 1 pts** part 3: wrong
- **- 2 pts** part 2: wrong set up and wrong u_2, u_3
- **- 1 pts** part 2: computational error
- **- 0.5 pts** Part 2 wrong, but did the correct thing in part 3
- **- 1.5 pts** part 2: they are orthogonal, but not normal
- **- 1 pts** part 3: missing
- **- 1 pts** part 4: missing
- **- 0.5 pts** part 4: half missing

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1.3

$A = \begin{pmatrix} -3 & -3 & 5 \\ 0 & -2 & 3 \\ 5 & 2 & -4 \end{pmatrix}$

1) $\left(\begin{array}{ccc|ccc} -3 & -3 & 5 & 1 & 0 & 0 \\ 0 & -2 & 3 & 0 & 1 & 0 \\ 5 & 2 & -4 & 0 & 0 & 1 \end{array}\right)$ $+\frac{5}{3}(i)$

$\left(\begin{array}{ccc|ccc} -3 & -3 & 5 & 1 & 0 & 0 \\ 0 & -2 & 3 & 0 & 1 & 0 \\ 0 & -3 & \frac{13}{3} & \frac{5}{3} & 0 & 1 \end{array}\right)$ $-\frac{3}{2}ii$, $-\frac{3}{2}ii$

$\left(\begin{array}{ccc|ccc} -3 & 0 & \frac{1}{2} & 1 & -\frac{3}{2} & 0 \\ 0 & -2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{6} & \frac{5}{3} & -\frac{3}{2} & 1 \end{array}\right)$ $+3iii$, $+18iii$

$\left(\begin{array}{ccc|ccc} -3 & 0 & 0 & 6 & -6 & 3 \\ 0 & -2 & 0 & 30 & -26 & 18 \\ 0 & 0 & -\frac{1}{6} & \frac{5}{3} & -\frac{3}{2} & 1 \end{array}\right)$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & 0 & -15 & 13 & -9 \\ 0 & 0 & 1 & -10 & 9 & -6 \end{array}\right)$

$\frac{5}{3} - \frac{12}{3} = \frac{13}{3}$

$\frac{5}{1} - \frac{9}{2} = \frac{10}{2} - \frac{9}{2}$

$\frac{13}{3} - \frac{9}{2} = \frac{26}{6} - \frac{27}{6} = -\frac{1}{6}$

$-\frac{3}{2} - \frac{9}{2} = -\frac{12}{2} = -6$

$\frac{5}{3} \cdot 18 = 30$

$1 + 27$

$\frac{5}{3} \cdot 6 =$

$-\frac{3}{2} \cdot 6 = -9$

$\boxed{A^{-1} = \begin{bmatrix} -2 & 2 & -1 \\ -15 & 13 & -9 \\ -10 & 9 & -6 \end{bmatrix}}$

2) $\det(A) = -3\left(\det\begin{pmatrix} -2 & 3 \\ 2 & -4 \end{pmatrix}\right) + 5\left(\det\begin{pmatrix} -3 & 5 \\ -2 & 3 \end{pmatrix}\right)$

$= -3\left(8 - 6\right) + 5\left(-9 + 10\right)$

$= -3(2) + 5$

$= -6 + 5 = \boxed{-1}$

3) $\boxed{\det(A_{11}) = 2}$ (from (2))

$\boxed{\det(A_{12}) = -15}$

$\boxed{\det(A_{13}) = 10}$

$\det(A_{21}) = (12) - 10 = \boxed{2}$

$\det(A_{22}) = 12 - 25 = \boxed{-13}$

$\det(A_{23}) = -6 - 15 = \boxed{9}$

$\det(A_{31}) = -9 + 10 = \boxed{1}$

$\det(A_{32}) = \boxed{-9}$

$\det(A_{33}) = \boxed{6}$

$\boxed{\operatorname{adj}(A) = \begin{bmatrix} 2 & -2 & 1 \\ 15 & -13 & 9 \\ 10 & -9 & 6 \end{bmatrix}}$

4) $A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}$

$= \frac{\begin{bmatrix} 2 & -2 & 1 \\ 15 & -13 & 9 \\ 10 & -9 & 6 \end{bmatrix}}{-1} = \begin{bmatrix} -2 & 2 & -1 \\ -15 & 13 & -9 \\ -10 & 9 & -6 \end{bmatrix}$

3 Problem 1.3 **5 / 5**

part 1

- **1 pts** Incorrect inverse
- **0.5 pts** Minor errors

part 2

- **1 pts** Incorrect determinant
- **0.5 pts** Minor errors

part 3

- **1 pts** Incorrect minors
- **1 pts** Incorrect adjoint
- **0.5 pts** Multiple minor errors
- **0.25 pts** One minor error

part 4

- **0.5 pts** Incorrect inverse
- **0.5 pts** No justification (no indication of the formula $$A^{-1}=\frac{1}{\det(A)} \text{adj}(A)$$).
- **0.25 pts** minor errors

✓ **- 0 pts** **Correct**

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1.4

1) $A = \begin{bmatrix} 1 & 24 & 24 \\ 0 & -2 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

1) $f_A(\lambda) = (1-\lambda)(-2-\lambda)(1-\lambda)$
$\boxed{= (1-\lambda)^2(-2-\lambda)}$

2) $\boxed{\begin{array}{l} \lambda = 1, -2 \\ \\ \text{almu}(1) = 2 \\ \text{almu}(-2) = 1 \end{array}}$

$-2-\lambda = 0$
$-\lambda = 2$
$\lambda = -2$

3) $\lambda = 1$:

$\left[\begin{array}{ccc|c} 0 & 24 & 24 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 0 & 24 & 24 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_2 + x_3 = 0 \quad x_2 = -x_3 = -u$
$x_1 = t$
$x_3 = u$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + u\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\boxed{\text{gemu}(1) = 2}$

$\lambda = -2$:

$\begin{array}{ccc} 3 & 24 & 24 \\ 0 & 0 & -3 \\ 0 & 0 & 3 \end{array}$

$\begin{array}{ccc} 3 & 24 & 24 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{array}$

rank = 2
dim ker = 1

$\boxed{\text{gemu}(-2) = 1}$

4) $\boxed{\begin{array}{l} \text{Eigenvalues of } A^5 \text{ are} \\ 1 \text{ and } -32 \\ \text{almu}(1) = 2,\ \text{gemu}(1) = 2 \\ \text{almu}(-32) = 1,\ \text{gemu}(-32) = 1 \end{array}}$

-2^5

$\boxed{\begin{array}{l} A \text{ is diagonalizable b/c} \\ \Sigma\,\text{almu} = \Sigma\,\text{gemu} \\ 3 = 3 \text{ and} \\ A \text{ is } 3 \times 3 \end{array}}$

4 Problem 1.4 5 / 5

✓ **- 0 pts** **Correct**

- 1 pts Characteristic polynomial

- 0.5 pts Eigenvalues

- 0.5 pts alg multiplicities

- 1 pts geometric mult.

- 1 pts diagonalizable?

- 1 pts fifth power

- 0.5 pts Partial credit on above.

1.5

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1) $\boxed{\begin{bmatrix}1 & 2\\0 & 3\end{bmatrix} = A}$

$\lambda = 1$, $\lambda = 3$

$\lambda = 1$: almu(1) = 1

$\begin{matrix}0 & 2\\0 & 2\\ \hline 0 & 1\\0 & 0\end{matrix}$ rank = 1 $\to$ dim ker = 1 $\to$ gemu(1) = 1

$\lambda = 3$: $\begin{matrix}-2 & 2\\0 & 0\\ \hline -1 & 1\\0 & 0\end{matrix}$ rank = 1 $\to$ dim ker = 1, gemu(3) = 1

almu(3) = 1

Σ almu = Σ gemu

$\lambda = 2$, $\begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}$ is 2x2 matrix,

$1, 3 \in \mathbb{R}$,

$\therefore \begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}$ diagonalizable over $\mathbb{R}$

2) $\boxed{A = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}}$

$\lambda = 0$ almu(0) = 2

gemu(0) = 1 (rank = 1)

$\therefore$ A not diagonalizable over $\mathbb{R}$

$A^2 = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$\lambda = 0$ almu(0) = 2

gemu(0) = 2 (rank = 0)

$\therefore A^2$ diagonalizable over $\mathbb{R}$ since gemu = almu and A is 2x2

3) $\boxed{\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} = A}$

$\lambda^2 + 1 = 0$

$\lambda^2 = -1$

$\lambda = \pm i$; $\boxed{\lambda \notin \mathbb{R}}$

$\lambda = i$:

$\begin{matrix}-i & -1\\1 & -i\\ \hline 1 & -i\\1 & -i\\ \hline 1 & -i\\0 & 0\end{matrix}$

$x_1 - i x_2 = 0$

$x_1 = -it$

$x_2 = t$

$\begin{pmatrix}x_1\\x_2\end{pmatrix} = t\begin{pmatrix}-i\\1\end{pmatrix}$ gemu = 1

$\lambda = -i$: $\begin{matrix}i & -1\\1 & i\\ \hline i & -1\\0 & 0\end{matrix}$

$\begin{pmatrix}x_1\\x_2\end{pmatrix} = t\begin{pmatrix}i\\-1\end{pmatrix}$ gemu = 1

Σ gemu = 2 = 2 ✓

4) $\boxed{A = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}}$ $\lambda^2 = 0$

$\lambda = 0 = 0 + 0i$

Similar rationale as (2), see demonstration of (2) not being diagonalizable over $\mathbb{R}$. The explanation holds for $\mathbb{C}$.

5 Problem 1.5 **5 / 5**

part 1

- **- 0.25 pts** Unclear justification, but correct example
- **- 0.5 pts** Incorrect example, but good attempt
- **- 1.25 pts** Incorrect

part 2

- **- 0.25 pts** Unclear justification, but correct example
- **- 0.5 pts** Incorrect example, but good attempt
- **- 1.25 pts** Incorrect

part 3

- **- 0.25 pts** Unclear justification, but correct example.
- **- 0.5 pts** Incorrect example, but good attempt
- **- 1.25 pts** Incorrect

part 4

- **- 0.25 pts** Unclear justification, but correct example
- **- 0.5 pts** Incorrect example, but good attempt
- **- 1.25 pts** Incorrect

✓ **- 0 pts** Correct

Nicholas [illegible]
[illegible]

1.6

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix}$$

1)
$$\begin{matrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{matrix}$$

$$(2-\lambda)(2-\lambda)^2 + 1(-2+\lambda)$$

$$\boxed{(2-\lambda)^3 + \lambda - 2 = f_A(\lambda)}$$

$$(2-\lambda)^3 + \lambda - 2 = 0$$
$$(2-\lambda)^3 + \lambda = 2$$

$\boxed{\lambda = 1} \rightarrow (2-1)^3 + 1 = 2$

~~$\lambda = 0 \rightarrow (2)^3$~~

~~$\lambda = -1 \rightarrow (3)^3$~~

$\boxed{\lambda = 2} \rightarrow (2-2) + 2 = 2$

$\boxed{\lambda = 3} \rightarrow (2-3)^3 + 3 = -1 + 3 = 2$

2) $\lambda = 1$:

$$\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix} \qquad \begin{matrix} x_1 = -x_3 \\ x_2 = 0 \\ x_3 = t \end{matrix}$$

$$\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{E_1 = \left\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}}$$

$\lambda = 2$

$$\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix}$$

$x_3 = 0$
$x_1 = 0$
$x_2 = t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\boxed{E_2 = \left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}}$$

$\lambda = 3$

$$\begin{matrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{matrix}$$

$$\begin{matrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} x_1 = x_3 \\ x_2 = 0 \\ x_3 = t \end{matrix}$$

$$\begin{matrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{E_3 = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}}$$

3) Basis
$$\left\{ \overset{v_1}{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \overset{v_3}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} \right\}$$

$v_1 \cdot v_2 = 0 + 0 + 0 = 0 \quad v_1 \perp v_2$

$v_1 \cdot v_3 = -1 + 0 + 1 = 0 \quad v_1 \perp v_3$

$v_2 \cdot v_3 = 0 + 0 + 0 = 0 \quad v_2 \perp v_3$

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad u_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Orthonormal basis $= \{u_1, u_2, u_3\}$

$$S = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad S^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \text{Recall } A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\boxed{A = SDS^T}$$

I'm sorry for being lazy but it's approaching 3hrs and my hand hurts ;-;

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4) $AS = SD$

$A = SDS^T$

$$A^n = (SDS^T)^n = SD^nS^T$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\frac{2^n}{\sqrt{2}} = \frac{2^n\sqrt{2}}{2} = \frac{2^{n-\frac{1}{2}}}{2}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{3^n}{\sqrt{2}} \\ 0 & 2^n & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{3^n}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1^n & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+\frac{3^n}{2} & 0 & -\frac{1}{2}+\frac{3^n}{2} \\ 0 & 2^n & 0 \\ -\frac{1}{2}+\frac{3^n}{2} & 0 & \frac{1}{2}+\frac{3^n}{2} \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} \frac{3^n+1}{2} & 0 & \frac{3^n-1}{2} \\ 0 & 2^n & 0 \\ \frac{3^n-1}{2} & 0 & \frac{3^n+1}{2} \end{bmatrix}}$$

6 Problem 1.6 **5 / 5**

part 1

- **0.5 pts** incorrect or incomplete list of eigenvalues
- **1 pts** incorrect characteristic polynomial

part 2

- **0.5 pts** one incorrect basis
- **1 pts** two incorrect bases
- **1.5 pts** three incorrect bases
- **1.5 pts** missing

part 3

- **0.25 pts** does not recognise non-orthonormality or non-basis (if earlier error was made)
- **0.5 pts** incorrect orthonormal basis
- **0.5 pts** incorrect orthogonal diagonalisation
- **1 pts** missing

part 4

- **0.25 pts** indicated general calculation but only computed (up to taking the n-th power of the diagonal) for a specific value of n
- **0.25 pts** did not write down an expression for the n-th power of the relevant diagonal matrix
- **0.25 pts** finds the correct pattern but does not fully justify it or obtain a closed-form result
- **0.5 pts** did not find a correct non-trivial expression for A^n
- **0.5 pts** insufficient or incorrect justification
- **0.75 pts** only computed for a specific value of n without correct general calculation
- **1 pts** missing

✓ **- 0 pts Correct**

- 2 pts Modifies the original matrix to a non-similar matrix in a way that leads to non-existence of an orthogonal diagonalisation