MATH 33A – SECTION 2
MIDTERM #2

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Discussion Section	2B

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Problem 1. Find $\text{proj}_V(\vec{x})$, where $\vec{x} = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and V is the image of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ given by $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x - 2y \\ 2x + 2y + 4z \\ x + z \\ y + z \end{bmatrix}$.

matrix assoc. w/ T: $A = \begin{array}{c} \begin{bmatrix} 2 & -2 & 0 \\ 2 & 2 & 4 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ \begin{array}{ccc} \| & \| & \| \\ \vec{v_1} & \vec{v_2} & \vec{v_3} \end{array} \end{array}$

$\vec{v_1} \cdot \vec{v_2} = 0 \Rightarrow \vec{v_1} \perp \vec{v_2}$ $\quad \|\vec{v_1}\| = \sqrt{2^2+2^2+1^2} = 3 = \|\vec{v_2}\|$

$\therefore \vec{u_1} = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u_2} = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

$\vec{v_3}^{\perp} = \vec{v_3} - (\vec{u_1} \cdot \vec{v_3})\vec{u_1} = \begin{bmatrix} 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3}\left(\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}\right) \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{9}(9)\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \quad \|\vec{v_3}^{\perp}\| = \sqrt{(-2)^2+2^2} = 2\sqrt{2}$

$\therefore \vec{u_3} = \frac{1}{2\sqrt{2}}\begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$

$\text{proj}_V(\vec{x}) = (\vec{u_1} \cdot \vec{x})\vec{u_1} + (\vec{u_2} \cdot \vec{x})\vec{u_2} + \cancel{(\vec{u_3} \cdot \vec{x})\vec{u_3}}$

$= \frac{1}{3}(9(2)) \cdot \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{3}(9(-2)) \cdot \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix} + 9\left(-\frac{1}{\sqrt{2}}\right)\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$

$= 2\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \frac{9}{\sqrt{2}}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25/2 \\ -9/2 \\ 2 \\ -2 \end{bmatrix}$

$\vec{v_1}$ $\qquad$ $\vec{v_2}$ $\qquad$ $\vec{v_3}$

Problem 2. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying

$$T\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\0\end{bmatrix}, \quad T\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

a)

(a) Find the matrix[1] A associated with T.

(b) Show that T is orthogonal.

$T(\vec{v_1}) = \begin{bmatrix}-1\\1\\0\end{bmatrix} = \vec{v_2} \Rightarrow [\vec{v_1}]_\beta = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\quad T(\vec{v_2}) = \vec{v_1} \Rightarrow [\vec{v_2}]_\beta = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$T(\vec{v_3}) = \vec{v_3} \Rightarrow [\vec{v_3}]_\beta = \begin{bmatrix}0\\0\\1\end{bmatrix}$ $\therefore B = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$

$S = \begin{bmatrix}1&-1&0\\1&1&0\\0&0&1\end{bmatrix}$ $\quad S^{-1}: \left[\begin{array}{ccc|ccc}1&-1&0&1&0&0\\1&1&0&0&1&0\\0&0&1&0&0&1\end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|ccc}1&-1&0&1&0&0\\0&2&0&-1&1&0\\0&0&1&0&0&1\end{array}\right]$

$\begin{array}{l}R_1+R_2\\ \frac{1}{2}R_2\end{array} \left[\begin{array}{ccc|ccc}1&0&0&1/2&1/2&0\\0&1&0&-1/2&1/2&0\\0&0&1&0&0&1\end{array}\right]$ $= S^{-1}$

$A = SBS^{-1}$ $\quad SB = \begin{bmatrix}1&-1&0\\1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} = \begin{bmatrix}-1&1&0\\1&1&0\\0&0&1\end{bmatrix}$

$SBS^{-1} = \begin{bmatrix}-1&1&0\\1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1/2&1/2&0\\-1/2&1/2&0\\0&0&1\end{bmatrix} = \begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix} = A$

$\begin{array}{ccc}\| & \| & \|\\ \vec{c_1} & \vec{c_2} & \vec{c_3}\end{array}$

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b) $\vec{c_1}\cdot\vec{c_2} = 0$, $\vec{c_1}\cdot\vec{c_3} = 0$, $\vec{c_2}\cdot\vec{c_3} = 0 \Rightarrow \vec{c_1} \perp \vec{c_2} \perp \vec{c_3}$

and $\|\vec{c_1}\| = \|\vec{c_2}\| = \|\vec{c_3}\| = 1$, $\therefore$ A orthogonal $\Rightarrow$ T orthogonal.

[1]Recall that if $\beta = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of $\mathbb{R}^3$ and B is the associated β-matrix of T, then $A = SBS^{-1}$, where $S = \begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix}$.

Problem 3. Compute det $\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix}}_{A}$.

$$\begin{array}{l} \\ R_2-R_1 \\ R_3-R_1 \\ R_4-R_1 \\ R_5-R_1 \end{array}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 3 & 9 & 19 & 34 \\ 0 & 4 & 14 & 34 & 69 \end{bmatrix} \begin{array}{l} \\ \\ R_3-2R_2 \\ R_4-3R_2 \\ R_5-4R_2 \end{array}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 3 & 10 & 22 \\ 0 & 0 & 6 & 22 & 53 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_4-3R_3 \\ R_5-6R_3 \end{array}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 4 & 17 \end{bmatrix}$$

$$\begin{array}{l} \\ \\ \\ \\ R_6-4R_5 \end{array}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \therefore \det A = 1^5 = 1$$

Problem 4. Determine whether the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is diagonalizable; in the affirmative case, give an invertible matrix S such that $S^{-1}AS = B$ is diagonal.

$$f_A(\lambda) = \det\begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix} = -\lambda \det\begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = \lambda^2 \det\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

$$= \lambda^2(\lambda^2 - 1) = 0 \quad \therefore \lambda = \begin{cases} 0 & \text{w/ alg mult } 2 \\ -1 & \text{" \quad " \quad } 1 \\ 1 & \text{" \quad " \quad } 1 \end{cases}$$

$$E_0 = \ker A = \ker\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

geo mult $= 2 =$ alg mult

$$E_1 = \ker(A - I_4) = \ker\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right)$$

geo mult $= 1 =$ alg mult

$$E_{-1} = \ker(A + I_4) = \ker\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}\right)$$

geo mult $= 1 =$ alg mult $\Longrightarrow$ $\therefore$ A is diagonalizable.

$$\therefore S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\underbrace{\quad\quad}_{0} \quad \underbrace{\quad}_{1} \quad \underbrace{\quad}_{-1}$$

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