MATH 33A – SECTION 2
MIDTERM #2

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Discussion Section	2 A

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Problem 1. Find $\text{proj}_V(\vec{x})$, where $\vec{x} = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and V is the image of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ given by $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x - 2y \\ 2x + 2y + 4z \\ x + z \\ y + z \end{bmatrix}$.

$$A = \begin{bmatrix} 2 & -2 & 0 \\ 2 & 2 & 4 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 0 \\ 2 & 2 & 4 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \\ 0 & 1 & \\ 0 & 1 & \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

basis of $\text{im}(a) = \left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ $\quad \|\vec{v}_1\| = \sqrt{4+4+1} = 3$

$\vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{3} \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

$\vec{v}_2^\perp = \vec{v}_1 - (\vec{u}_1 \cdot \vec{v}_1)\, \vec{u}_1$
$= \vec{v}_2 - \frac{1}{3}(0)\,\vec{u}_1 = \vec{v}_2$

$\text{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\,\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\,\vec{u}_2 = 9 \cdot \frac{2}{3} \cdot \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + 9 \cdot \frac{-2}{3} \cdot \frac{1}{3} \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix} =$

$$\begin{bmatrix} 4 - (-4) \\ 4 - 4 \\ 2 - 0 \\ 0 - 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 2 \\ -2 \end{bmatrix}$$

Problem 2. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying

$$T\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\0\end{bmatrix}, \quad T\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

(a) Find the matrix[1] A associated with T.
(b) Show that T is orthogonal.

a) $\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}a+b=-1\\ d+e=1\\ g+h=0\end{bmatrix}$ $\quad \begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}-a+b=1\\ -d+e=1\\ -g+h=0\end{bmatrix}$

$\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}c=0\\ f=0\\ i=1\end{bmatrix}$

$g+h=0$
$-g+h=0$
$2h=0$
$g=0$

$-a+b=-1$
$-a+b=1$
$2b=0$
$a=-1$

$d+e=1$
$-d+e=1$
$2e=2 \quad e=1$
$d=0$

$A = \begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

b) $\overset{r_1}{\begin{bmatrix}-1\\0\\0\end{bmatrix}} \cdot \overset{r_2}{\begin{bmatrix}0\\1\\0\end{bmatrix}} = 0 \qquad \overset{r_2}{\begin{bmatrix}0\\1\\0\end{bmatrix}} \; \overset{r_3}{\begin{bmatrix}0\\0\\1\end{bmatrix}}$

$\overset{r_1}{\begin{bmatrix}-1\\0\\0\end{bmatrix}} \cdot \overset{r_3}{\begin{bmatrix}0\\0\\1\end{bmatrix}} = 0$

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[1] Recall that if $\beta = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of $\mathbb{R}^3$ and B is the associated β-matrix of T, then $A = SBS^{-1}$, where $S = \begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix}$.

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Problem 3. Compute det $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix}$.

$$\det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 3 & 9 & 19 & 34 \\ 0 & 4 & 14 & 34 & 69 \end{bmatrix} = 1\cdot\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 9 & 14 \\ 3 & 9 & 19 & 34 \\ 4 & 14 & 34 & 69 \end{bmatrix} = \det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 6 \\ 0 & 3 & 10 & 22 \\ 0 & 6 & 22 & 53 \end{bmatrix} =$$

$$\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 4 & 17 \end{bmatrix} = \det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1$$

$69-16=53$

$53-36=17$

Problem 4. Determine whether the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is diagonalizable; in the affirmative case, give an invertible matrix S such that $S^{-1}AS = B$ is diagonal.

$$\det \begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix} = (-\lambda)\det \begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = (-\lambda)^2 \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} =$$

$$\left((-\lambda)^2 - 1\right)(-\lambda)^2 = (-\lambda)^2(-\lambda+1)(-\lambda-1)$$

eigenvalues $\begin{cases} \lambda = 0 & \text{with al. mult} = 2 \\ \lambda = 1 & \text{'' '' '' } = 1 \\ \lambda = -1 & \text{'' '' '' } = 1 \end{cases}$

$$E_0 = \ker \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ geo mult} = 2$$

$$E_1 = \ker \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ geo mult} = 1$$

$$E_{-1} = \ker \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\} \text{ geomult} = 1$$

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$\underbrace{\quad\quad}_{E_0}$ $\underbrace{\quad}_{E_1}$ $\underbrace{\quad}_{E_{-1}}$

✓ geo mult = alg. mult for all eigenvalues of A, so A is diagonalizable