MATH 33A – SECTION 2
MIDTERM #1

OCTOBER 23, 2015

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Discussion Section	2C

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Problem 1. Determine whether there exists a polynomial of the form

$$f(t) = a + bt + ct^2 + dt^3$$

whose graph goes through the points $(0, 1)$, $(1, 0)$, $(-1, 0)$, $(2, -15)$, $(-2, 9)$ and $(3, -56)$. In the affirmative case, give $f(t)$.

$f(0) = a = 1$, $f(1) = a + b + c + d = 0$

$f(-1) = a - b + c - d = 0$, $f(2) = a + 2b + 4c + 8d = -15$

$f(-2) = a - 2b + 4c - 8d = 9$, $f(3) = a + 3b + 9c + 27d = -56$

$$A' = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 1 & 2 & 4 & 8 & -15 \\ 1 & -2 & 4 & -8 & 9 \\ 1 & 3 & 9 & 27 & -56 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \\ R_5 - R_1 \\ R_6 - R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & -1 & -1 \\ 0 & 2 & 4 & 8 & -16 \\ 0 & -2 & 4 & -8 & 8 \\ 0 & 3 & 9 & 27 & -57 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ R_5 + R_4 \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & -1 & -1 \\ 0 & 2 & 4 & 8 & -16 \\ 0 & 0 & 8 & 0 & -8 \\ 0 & 3 & 9 & 27 & -57 \end{array}\right] \begin{array}{l} \\ \\ R_3 + R_2 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 2 & 4 & 8 & -16 \\ 0 & 0 & 8 & 0 & -8 \\ 0 & 3 & 9 & 27 & -57 \end{array}\right] \begin{array}{l} \\ \\ \tfrac{1}{2}R_3 \\ \\ \tfrac{1}{8}R_5 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 2 & 4 & 8 & -16 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 3 & 9 & 27 & -57 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ R_5 - R_3 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 2 & 4 & 8 & -16 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 9 & 27 & -57 \end{array}\right] \begin{array}{l} \\ \\ \\ \tfrac{1}{2}R_4 \\ \\ \tfrac{1}{3}R_6 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 9 & -19 \end{array}\right] \begin{array}{l} \\ R_2 - R_3 \\ \\ \\ \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 9 & -19 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - R_2 \\ \\ R_6 - R_2 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 3 & -8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 8 & -19 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - 2R_3 \\ \\ R_6 - 3R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 3 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & -16 \end{array}\right] \begin{array}{l} \\ \\ \\ \tfrac{1}{3}R_4 \\ \\ \tfrac{1}{8}R_6 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{l} \\ R_2 - R_4 \\ \\ R \\ \\ R_6 - R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{f(t) = 1 + 2t - t^2 - 2t^3}$$

Problem 2. Consider the matrix

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Recall that the linear transformation $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $R_\theta(\vec{x}) = A_\theta\vec{x}$ is a counter-clockwise rotation through an angle θ.

15 (a) Show that

$$R_\alpha \circ R_\beta = R_\beta \circ R_\alpha = R_{\alpha+\beta}$$

by calculation.

15 (b) Using the result of (a), show that R_α is invertible and describe R_α^{-1}.

Hint: For (a), you may use the trigonometric identities

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

a. $[R_\alpha] = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ $[R_\beta] = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$

$[R_\alpha \circ R_\beta] = [R_\alpha] \cdot [R_\beta] = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cancel{\cos\alpha\cos\beta + \sin\alpha\sin\beta}\ -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix}$

$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$

$[R_\beta \circ R_\alpha] = [R_\beta] \circ [R_\alpha] = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix}$

$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$

$= [R_\alpha \circ R_\beta] = [R_{\alpha+\beta}]$, $\therefore$

$\boxed{R_\alpha \circ R_\beta = R_\beta \circ R_\alpha = R_{\alpha+\beta}}$

$\longrightarrow$

b. From (a), $R_\alpha \circ R_{-\alpha} = R_{\alpha-\alpha} = R_0$

$$[R_0] = \begin{bmatrix} \cos(0) & -\sin(0) \\ \sin(0) & \cos(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \therefore$$

R_α is invertible and $R_\alpha^{-1} = R_{-\alpha}$.

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Problem 3. Determine all the values (if any) of the constants B and C for which the following matrix is invertible:

$$A = \begin{bmatrix} 0 & 1 & B \\ -1 & 0 & C \\ -B & -C & 0 \end{bmatrix}$$

$$\operatorname{rref}[A : I_3] = \left[\begin{array}{ccc|ccc} 0 & 1 & B & 1 & 0 & 0 \\ -1 & 0 & C & 0 & 1 & 0 \\ -B & -C & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} -1 & 0 & C & 0 & 1 & 0 \\ 0 & 1 & B & 1 & 0 & 0 \\ -B & -C & 0 & 0 & 0 & 1 \end{array}\right] -R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -C & 0 & -1 & 0 \\ 0 & 1 & B & 1 & 0 & 0 \\ -B & -C & 0 & 0 & 0 & 1 \end{array}\right] R_3 - R_1 - R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -C & 0 & -1 & 0 \\ 0 & 1 & B & 1 & 0 & 0 \\ -B-1 & -C-1 & C-B & -1 & 1 & 1 \end{array}\right]$$

A invertible if $\begin{bmatrix} 1 & 0 & -C \\ 0 & 1 & B \\ -B-1 & -C-1 & C-B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

why?

$-C = 0$

$B = 0$

$-B - 1 = 0 \rightarrow 0 - 1 = 0$ inconsistent

$-C - 1 = 0$

$C - B = 1$

Problem 4. Let $A = \begin{bmatrix} 4 & 8 & 1 & 1 & 4 \\ 3 & 6 & 1 & 2 & 5 \\ 2 & 4 & 1 & 9 & 10 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix}$.

(a) Find a basis for ker(A).

(b) Find a basis for im(A).

a.
$$\left[\begin{array}{ccccc:c} 4 & 8 & 1 & 1 & 4 & 0 \\ 3 & 6 & 1 & 2 & 5 & 0 \\ 2 & 4 & 1 & 9 & 10 & 0 \\ 1 & 2 & 3 & 2 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - 4R_4 \\ R_2 - 3R_4 \\ R_3 - 2R_4 \\ \\ \end{array} \left[\begin{array}{ccccc:c} 0 & 0 & -11 & -7 & 4 & 0 \\ 0 & 0 & -8 & -4 & 5 & 0 \\ 0 & 0 & -5 & 5 & 10 & 0 \\ 1 & 2 & 3 & 2 & 0 & 0 \end{array}\right] R_3 - 2R_2$$

$$\begin{bmatrix} 0 & 0 & -11 & -7 & 4 & 0 \\ 0 & 0 & -8 & -4 & 5 & 0 \\ 0 & 0 & 11 & 13 & 0 & 0 \\ 1 & 2 & 3 & 2 & 0 & 0 \end{bmatrix} R_1 + R_3 \begin{bmatrix} 0 & 0 & 0 & 6 & 4 & 0 \\ 0 & 0 & -8 & -4 & 5 & 0 \\ 0 & 0 & 11 & 13 & 0 & 0 \\ 1 & 2 & 3 & 2 & 0 & 0 \end{bmatrix} R_3 + R_2 \begin{bmatrix} 0 & 0 & 0 & 6 & 4 \\ 0 & 0 & -8 & -4 & 5 \\ 0 & 0 & 3 & 9 & 5 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} \tfrac{1}{2}R_1$$

$$\begin{bmatrix} 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & -8 & -4 & 5 \\ 0 & 0 & 3 & 9 & 5 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} R_3 - 2R_1 \begin{bmatrix} 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & -8 & -4 & 5 \\ 0 & 0 & 3 & 3 & 1 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} R_2 - 2R_1 - R_3 \begin{bmatrix} 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & -11 & -13 & 0 \\ 0 & 0 & 3 & 3 & 1 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} R_2 + 3R_3$$

$$\begin{bmatrix} 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 3 & 3 & 1 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} R_3 - R_2 \begin{bmatrix} 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 2 & 4 & -2 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} \tfrac{1}{2}R_3 \begin{bmatrix} 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 2 & -1 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} R_2 - R_3 \begin{bmatrix} 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & -3 & 4 \\ 0 & 0 & 1 & 2 & -1 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} R_2 + R_1$$

$$\begin{bmatrix} 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 2 & -1 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} \begin{array}{l} \tfrac{1}{6}R_2 \\ R_3 - R_1 \end{array} \begin{bmatrix} 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -3 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} \tfrac{1}{3}R_1, -2R_2 \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -3 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} R_3 + R_1 + 3R_2 \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} R_4 - 3R_3$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

ker?

b. basis im$(A) = \text{span}\left(\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \ldots\right.$

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