Math 33A-1 Midterm 2 Summer 2020

For the instructions for this test, see the dedicated PDF.

1. Consider the following vectors in $\mathbb{R}^2$: $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Consider the two bases $\mathcal{B}_1 = (\vec{e}_1, \vec{e}_2)$ and $\mathcal{B}_2 = (\vec{v}_1, \vec{v}_2)$.

   (a) **(3 pts)** Find the matrix S of change of coordinates from $\mathcal{B}_2$ to $\mathcal{B}_1$ (i.e., turning $\mathcal{B}_2$-coordinates into $\mathcal{B}_1$-coordinates).

   (b) **(3 pts)** Find the matrix of change of coordinates from $\mathcal{B}_1$ to $\mathcal{B}_2$ (i.e., turning $\mathcal{B}_1$-coordinates into $\mathcal{B}_2$-coordinates).

   (c) **(3 pts)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with $\mathcal{B}_1$-matrix $A = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$. Find the $\mathcal{B}_2$-matrix B of the transformation T.

   (d) **(3 pts)** Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with $F(\vec{v}_1) = 3\vec{v}_1 - \vec{v}_2$ and $F(\vec{v}_2) = \vec{v}_1 + 2\vec{v}_2$. Find the $\mathcal{B}_2$-matrix of the transformation F.

2. **(8 pts)** Compute the determinant of the following matrix. You can use any method you want, but you need to motivate and show work for each step you perform.

$$A = \begin{pmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix}.$$

3. Let V be the 2-plane in $\mathbb{R}^4$ defined by the following equations:

$$\begin{cases} x_1 + x_2 + x_3 \phantom{{}+x_4} = 0 \\ \phantom{x_1 + {}} x_2 + x_3 + x_4 = 0 \end{cases}$$

   That is, a vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ is in V if and only if its coordinates are a solution of the above linear system.

   (a) **(4 pts)** Find a basis of V.

   (b) **(4 pts)** Find an orthonormal basis of V.

   (c) **(4 pts)** Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the orthogonal projection onto V. Determine the matrix of T. If your answer arises as the product of two (or more) matrices, you can write your solution as a product of matrices, and you do not have to compute the product explicitly. On the other hand, each matrix appearing in the product has to be explicitly given (i.e., if you need the inverse or transpose of some matrix, you need to compute it).

(d) **(2 pts)** Does $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ belong to $V^{\perp}$? For full credit, explain your reasoning (a computation may suffice).

4. (**6 pts**) Consider the following vectors:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Find a fourth vector $\vec{v}_4$ so that the 4-volume of the 4-parallelepyped generated by $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ is 1. You need to show work motivating your choice of $\vec{v}_4$. Note that checking that the vector you exhibit works does not provide any evidence as for why you chose the vector. If your work starts from a wrong attempt that then indicates you the solution, please include the wrong attempt too (specifying that it is wrong).

5. Let A be an $n \times n$ matrix, and assume that $\vec{v}$ is an eigenvector of A with eigenvalue 2.

(a) **(2 pts)** Is $\vec{v}$ an eigenvector of the matrix A^2? If so, what is the corresponding eigenvalue? Motivate your answer.

(b) **(2 pts)** Is $\vec{v}$ an eigenvector of the matrix $A^2 + I_n$? If so, what is the corresponding eigenvalue? Motivate your answer.

(c) **(2 pts)** Let S be an invertible $n \times n$ matrix, and let $\vec{w}$ be the vector such that $\vec{v} = S\vec{w}$. Is $\vec{w}$ an eigenvector of $S^{-1}AS$? If so, what is the corresponding eigenvalue? Motivate your answer.

6. For each of the following cases, provide an example satisfying the stated property, or state why it is impossible.

(a) **(2 pts)** Two 4×4 matrices A and B so that $\text{rank}(A) = 3$, $\text{rank}(B) = 2$, and $\text{rank}(AB) = 0$.

(b) **(2 pts)** A 2×2 matrix A so that A is not symmetric but A^2 is symmetric.