Math 33A-1 Midterm 2 Fall 2019

Name: ______________________ uid: ______________________

Section: ___________ Signature: ___________________________

Instructions:

- Unless otherwise stated, you need to justify your answer. Please show all of your work, as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown.
- You have 50 minutes to complete the exam.
- All answers should be completely simplified, unless otherwise stated.
- This is a closed book and closed notes test. You may **not** use a scientific calculator. No electronics are allowed on this exam. Make sure all cell phones are silenced, put away and out of sight. If you have a cell phone out at any point, for any reason, this will constitute a violation of test policy, and you may receive a zero on this exam.
- If asked, you must show us your **bruin card**.
- You may ask for scratch paper. You may use **no** other scratch paper. Please transfer all finished work onto the proper page in the test for us to grade there. We will **not** grade the work on the scratch page.
- Notice that the test is printed just front, so the space left for each question is sufficient, but possibly not necessary, to answer the questions. If you write on the back of a page, please indicate it.

STUDENT: PLEASE DO NOT WRITE BELOW THIS LINE. THIS TABLE IS TO BE USED FOR GRADING.

Problem	Points	Score
1	12	12
2	10	10
3	12	12
4	6	6
5	4	4
6	6	6
Total	50	50

nice!

1. Consider the following vectors in $\mathbb{R}^2$: $\vec{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$, $\vec{v}_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}1\\1\end{pmatrix}$.
Consider the two bases $\mathcal{B}_1 = (\vec{e}_1, \vec{e}_2)$ and $\mathcal{B}_2 = (\vec{v}_1, \vec{v}_2)$. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by $T(\vec{v}_1) = 2\vec{v}_1 + 3\vec{v}_2$ and $T(\vec{v}_2) = 2\vec{v}_2$.

S goes for B-coord to std

B_1 = standard $\quad B_2 = \left(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right)$

(a) (4 pts) Find the matrix of change of coordinates from $\mathcal{B}_2$ to $\mathcal{B}_1$ (i.e., turning $\mathcal{B}_2$-coordinates into $\mathcal{B}_1$-coordinates).

B to std

$$S = (\vec{v_1} \mid \vec{v_2}) = \boxed{\begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}}$$

4/4

(b) (4 pts) Find the matrix of change of coordinates from $\mathcal{B}_1$ to $\mathcal{B}_2$ (i.e., turning $\mathcal{B}_1$-coordinates into $\mathcal{B}_2$-coordinates).

$$S^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix} = \frac{1}{(1)(1)-(1)(2)}\begin{pmatrix}1 & -1\\ -2 & 1\end{pmatrix} = \boxed{\begin{pmatrix}-1 & 1\\ 2 & -1\end{pmatrix}}$$

$1-2$
-1

4/4

(c) (4 pts) Find the $\mathcal{B}_2$-matrix of T.

$$B_2\text{-matrix} = \left((T(\vec{v_1}))_{B_2} \mid (T(\vec{v_2}))_{B_2} \right)$$

$T(\vec{v_1}) = 2\vec{v_1} + 3\vec{v_2} \quad \begin{pmatrix}2\\3\end{pmatrix}$

$T(\vec{v_2}) = 2\vec{v_2} \quad \begin{pmatrix}0\\2\end{pmatrix}$

$$= \boxed{\begin{pmatrix}2 & 0\\ 3 & 2\end{pmatrix}}$$

4/4

2. Consider the following vectors in $\mathbb{R}^3$: $\vec{v_1} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and let $V = \text{span}(\vec{v_1}, \vec{v_2})$.

(a) **(4 pts)** Perform the Gram–Schmidt process on $(\vec{v_1}, \vec{v_2})$ to find an orthonormal basis for V.

$$\vec{u_1} = \frac{1}{\|\vec{v_1}\|}\vec{v_1} = \frac{1}{2}\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\|\vec{v_1}\| = \sqrt{2^2} = \sqrt{4} = 2$$

$$\cancel{1+0+0}\ \ 1$$

$$\vec{v_2}^{\perp} = \vec{v_2} - \vec{v_2}^{\parallel} = \vec{v_2} - (\vec{u_1}\cdot\vec{v_2})\vec{u_1} = \vec{v_2} - \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{u_2} = \frac{1}{\|\vec{v_2}^{\perp}\|}\vec{v_2}^{\perp} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\|\vec{v_2}^{\perp}\| = \sqrt{1+1} = \sqrt{2}$$

$$\boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}$$

✓ +4.

(b) **(4 pts)** Consider the vector $\vec{x} = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$. Compute $\text{proj}_V(\vec{x})$.

$$\text{proj}_V(\vec{x}) = (\vec{u_1}\cdot\vec{x})\vec{u_1} + (\vec{u_2}\cdot\vec{x})\vec{u_2}$$

$$\vec{u_1}\cdot\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix} = 5 \qquad \vec{u_2}\cdot\vec{x} = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\cdot\begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix} = \frac{4}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{6}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$

$$= 5\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3\sqrt{2}\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = \boxed{\begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}}$$

✓ +4

(c) **(2 pts)** Let $\vec{w}$ be a vector in $\mathbf{R}^3$ with $\|\vec{w}\| = 8$ and $\|\text{proj}_V(\vec{w})\| = 8$. Determine $\dim(\text{span}(\vec{v_1}, \vec{v_2}, \vec{w}))$. Motivate your answer.

if $\|\vec{w}\| = \|\text{proj}_V(\vec{w})\|$, then $\vec{w}$ is in the subspace V spanned by $\vec{v_1}, \vec{v_2}$.
In other words, it is redundant, so

$$\dim(\text{span}(\vec{v_1}, \vec{v_2}, \vec{w})) = \dim(\text{span}(\vec{v_1}, \vec{v_2})) = 2$$ ✓ +2

$4+4+2=10$

3. Compute the following determinants. You can use any method you want, but you need to motivate each step you perform.

(a) (8 pts) $\det\begin{pmatrix} 2 & 2 & 2 & 2 \\ 3 & 4 & 1 & 7 \\ 1 & 1 & 1 & 2 \\ 3 & 3 & 4 & 3 \end{pmatrix} = -\det \begin{array}{c} III \\ II \\ I \\ IV \end{array}\begin{pmatrix} 1 & 1 & 1 & 2 \\ 3 & 4 & 1 & 7 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 4 & 3 \end{pmatrix} = -\det \begin{array}{c} I \\ II-3I \\ III-2I \\ IV-3I \end{array}\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{pmatrix}$

$= \det \begin{array}{c} I \\ II \\ IV \\ III \end{array}\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & -2 \end{pmatrix} = -2\det\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -2(1) = \boxed{-2}$

$E_{2\cdot 1}$

8/8

$\det(AB) = \det(A)\det(B)$

(b) (4 pts) $\det\left(\begin{pmatrix} 28 & 25 \\ 15 & 17 \end{pmatrix}\begin{pmatrix} 1 & 25 \\ 3 & 100 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 25 & 14 \\ 3 & 17 \end{pmatrix}\right)$

$= \det\begin{pmatrix} 28 & 25 \\ 15 & 17 \end{pmatrix} \det\begin{pmatrix} 1 & 25 \\ 3 & 100 \end{pmatrix} \det\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \det\begin{pmatrix} 25 & 14 \\ 3 & 17 \end{pmatrix} = \boxed{0}$

$\det\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (0)0 - 1(0) = \underline{0}$

4/4

4. **(6 pts)** Let $T\colon \mathbb{R}^3 \to \mathbb{R}^1$ be the linear trasnformation given by

$$T(\vec{x}) = \det \begin{pmatrix} 1 & 0 & x_1 \\ 2 & -1 & x_2 \\ 3 & 5 & x_3 \end{pmatrix},$$

1
1
8

$(-8-5)+(10+3)$
$-13+13=0$

where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find a basis of $\ker(T)$. Motivate your answer.

$T(\vec{x}) = 0$ when one of the rows in the matrix is redundant, since the matrix will not be invertible

All linear combinations of the given columns are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix},$$

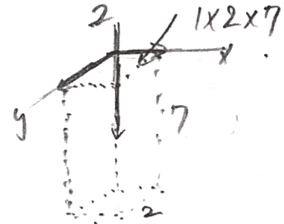
so basis of $\ker(T)$ is $\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix}\right)$

✓6/6

5. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix in the standard basis is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -7 \end{pmatrix}.$$

$1 \times 2 \times 7 = \underline{14}$



(a) **(2 pts)** Let Ω be a sphere of volume 5 in $\mathbb{R}^3$. Compute the volume of $T(\Omega)$. Motivate your answer.

$$|\det(A)| = \frac{\text{volume } T(\Omega)}{\text{volume }(\Omega)} \longrightarrow \text{volume } T(\Omega) = |\det(A)| \cdot \text{volume}(\Omega)$$

$\det(A) = 1 \cdot 2(-7) = -14$

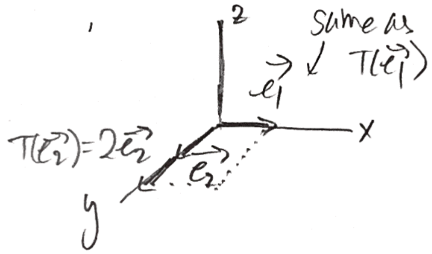
volume $T(\Omega) = |-14| \cdot 5 = 14 \cdot 5 = \boxed{70}$

✓ +2

$$\begin{array}{r} {}^{2}14 \\ \times\ 5 \\ \hline 70 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

(b) (2 pts) Let S be the square in $\mathbb{R}^3$ obtained as the 2-parallelepiped defined by $\vec{e}_1$ and $\vec{e}_2$ (i.e., the square with vertices $\vec{0}$, $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_1 + \vec{e}_2$). Determine the area of $T(S)$. Motivate your answer.



when looking at S, only the transformations applied to $\vec{e_1}, \vec{e_2}$ affect its area. Since $T(\vec{e_1}) = \vec{e_1}$ and $T(\vec{e_2}) = 2\vec{e_2}$ its area is scaled by $\|T(\vec{e_1})\|\|T(\vec{e_2})\| = 1(2) = 2$
the area of S is 1, so area $T(S) = 1 \cdot 2 = \boxed{2}$

+2

6. For each of the following cases, write matrices satisfying the stated property, or state why it is impossible.

2+2=4.

(a) (2 pts) A is a 3×3 invertible matrix such that A and A^{-1} have integer entries and $\det(A) = 4$.

Impossible: if both A and A^{-1} have integer entries, the det(A) must be 1 or -1.

v2/2

→ $\det(A) = 4$ means volume is scaled by 4, meaning its inverse would scale it by $1/4$, which requires non-integer values to have $\det(A^{-1}) = 1/4$.

(b) (2 pts) A a 3×2 matrix such that $\det(A^T A) \neq 0$.

$\boxed{A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}$ $A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\det(I_2) = 1$

v2/2

A^T 2x2 matrix
$\mathbb{R}^2 \to \mathbb{R}^3$

(c) (2 pts) A a 2×3 matrix such that $\det(A^T A) \neq 0$.

impossible: if A is a 2x3 matrix, the max rank is 2, or equivalently, $\dim(\text{im}(A)) = 2$.

applying A^T, $\mathbb{R}^2 \to \mathbb{R}^3$ cannot add dimensions to the image. And then $\dim(\text{im}(A^T A)) \leq 2 < n = 3$, so $A^T A$ is not invertible, meaning $\det(A^T A) = 0$ for all A.

v2/2