Math 33A-1 Midterm 1 — Summer 2020

For the instructions for this test, see the dedicated PDF.

1. **(6 pts)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformations determined by
$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}7\\3\end{pmatrix}, \quad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}, \quad S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}5\\1\end{pmatrix}, \quad S\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\4\end{pmatrix}.$$
Find the matrix associated to the linear transformation $S \circ T\colon \mathbb{R}^2 \to \mathbb{R}^2$.

2. **(6 pts)** Consider the vectors $\vec{a} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\vec{b} = \begin{pmatrix}-1\\1\end{pmatrix}$. Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(\vec{x}) = (\vec{x}\cdot\vec{a})\vec{b}$. Find the matrix associated to the linear transformation T.

3. Let A be the following matrix
$$A = \begin{pmatrix}1 & -2 & k\\ 2 & -5 & 3\\ -2 & 5 & -4\end{pmatrix},$$
where k is some parameter.
   (a) **(8 pts)** Using row operations (no credit otherwise!), find (if possible) the inverse matrix of A. (Hint: you can check your answer multiplying it by A)
   (b) **(3 pts)** Find all the solutions of the following linear system:
$$\begin{pmatrix}1 & -2 & k\\ 2 & -5 & 3\\ -2 & 5 & -4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix}.$$

4. Let A be the following matrix with the following reduced row-echelon form
$$A = \begin{pmatrix}3 & 0 & -3 & 12 & 0\\ 1 & 2 & 3 & 10 & 0\\ 2 & 1 & 0 & 11 & 1\\ 0 & 1 & 2 & 3 & 1\end{pmatrix}, \quad \operatorname{rref}(A) = \begin{pmatrix}1 & 0 & -1 & 4 & 0\\ 0 & 1 & 2 & 3 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}.$$
   (a) **(1 pt)** What is $\operatorname{rank}(A)$?
   (b) **(4 pts)** Find a basis of the kernel of A.
   (c) **(2 pts)** Can you find a vector $\vec{b}$ in $\mathbb{R}^4$ so that the linear system $A\vec{x} = \vec{b}$ has infinitely many solutions? If yes, provide an example, if not, motivate why.

(d) **(2 pts)** Can you find a vector $\vec{c}$ in $\mathbb{R}^4$ so that the linear system $A\vec{x} = \vec{c}$ has exactly one solution? If yes, provide an example, if not, motivate why.

5. Consider the following vectors in $\mathbb{R}^4$: $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ -2 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 3 \\ 6 \\ -3 \end{pmatrix}.$

   (a) **(5 pts)** Is the vector $\begin{pmatrix} 3 \\ 6 \\ 10 \\ -6 \end{pmatrix}$ in $\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$? If yes, exhibit this vector as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$, if not, explain why not by analyzing the associated set of linear equations. If your answer is affirmative, the choice of linear combination has to be motivated by proof of work.

   (b) **(5 pts)** Is the vector $\begin{pmatrix} 1 \\ 7 \\ 12 \\ 3 \end{pmatrix}$ in $\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$? If yes, exhibit this vector as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$, if not, explain why not by analyzing the associated set of linear equations. If your answer is affirmative, the choice of linear combination has to be motivated by proof of work.

6. For each of the following cases, provide an example satisfying the stated property, or state why it is impossible.

   (a) **(2 pts)** A 3×3 matrix A so that A^2 is invertible, but A is not invertible.

   (b) **(2 pts)** An invertible 2×2 matrix so that $A^{-1} = A$ and A is **not** diagonal.

   (c) **(2 pts)** A linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ so that $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, $T\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $T\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 21 \end{pmatrix}$.

   (d) **(2 pts)** A 2×2 matrix A with $\text{im}(A) = \text{span}\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ and $\ker(A) = \text{span}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.