Math 33A-1 Midterm 1 Fall 2019

Name: ______________________ uid: ______________________

Section: __________ Signature: ______________________

Instructions:

- Unless otherwise stated, you need to justify your answer. Please show all of your work, as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown.
- You have 50 minutes to complete the exam.
- All answers should be completely simplified, unless otherwise stated.
- This is a closed book and closed notes test. You may **not** use a scientific calculator. No electronics are allowed on this exam. Make sure all cell phones are silenced, put away and out of sight. If you have a cell phone out at any point, for any reason, this will constitute a violation of test policy, and you may receive a zero on this exam.
- If asked, you must show us your **bruin card**.
- You may ask for scratch paper. You may use **no** other scratch paper. Please transfer all finished work onto the proper page in the test for us to grade there. We will **not** grade the work on the scratch page.
- Notice that the test is printed just front, so the space left for each question is sufficient, but possibly not necessary, to answer the questions. If you write on the back of a page, please indicate it.

STUDENT: PLEASE DO NOT WRITE BELOW THIS LINE. THIS TABLE IS TO BE USED FOR GRADING.

Problem	Points	Score
1	8	8
2	4	1
3	7	7
4	4	4
5	4	4
6	15	15
7	8	8
Total	50	47

1. **(8 pts)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformations determined by

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}, \quad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}, \quad S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}, \quad S\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Find the matrix associated to the linear transformation $S \circ T\colon \mathbb{R}^2 \to \mathbb{R}^2$.

$A = \left(T\begin{pmatrix}1\\0\end{pmatrix}\ T\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}2 & 1\\1 & 0\end{pmatrix}$ $\quad B = \left(S\begin{pmatrix}1\\0\end{pmatrix}\ S\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}0 & 0\\-1 & 0\end{pmatrix}$

$S \circ T = BA = \begin{pmatrix}0 & 0\\-1 & 0\end{pmatrix}\begin{pmatrix}2 & 1\\1 & 0\end{pmatrix}$

$\begin{pmatrix}0 & 0\\-1 & 0\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = 2\begin{pmatrix}0\\-1\end{pmatrix} + 1\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\-2\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\-2\end{pmatrix}$

$\begin{pmatrix}0 & 0\\-1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 1\begin{pmatrix}0\\-1\end{pmatrix} + 0\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$

$BA = \begin{pmatrix}0 & 0\\-2 & -1\end{pmatrix}$ ✓

8/8

2. **(4 pts)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function with

$\begin{pmatrix}2\\1\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix} \begin{pmatrix}2\\2\end{pmatrix}$

$$T\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}8\\5\end{pmatrix}, \quad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}, \quad T\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}5\\5\end{pmatrix}.$$

Can T be a linear transformation? If yes, provide an example. If not, provide an explanation.

$\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}2\\2\end{pmatrix}$

$T\begin{pmatrix}1\\0\end{pmatrix} = T\begin{pmatrix}2\\1\end{pmatrix} - \frac{1}{2}T\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}8\\5\end{pmatrix} - \frac{1}{2}\begin{pmatrix}5\\5\end{pmatrix} = \begin{pmatrix}8\\5\end{pmatrix} - \begin{pmatrix}5/2\\5/2\end{pmatrix} = \begin{pmatrix}\frac{11}{2}\\\frac{5}{2}\end{pmatrix}$

$A = \left(T\begin{pmatrix}1\\0\end{pmatrix}\ T\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}\frac{11}{2} & 1\\\frac{5}{2} & 0\end{pmatrix}$ ← Yes ✗

1/4

3. For each of the following cases, write a matrix satisfying the stated property, or state why it is impossible.

(a) (2 pts) A is a 5×6 matrix of rank 6.

Impossible because the rank of the matrix has to be less than or equal to the the number of rows or columns, whichever is smaller. 6 is not less than or equal to 5.

2/2

(b) (2 pts) A is a 3×2 matrix of rank 1.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

2/2

(c) (3 pts) A is a 3×3 matrix with $\ker(A) = \text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$. $\begin{pmatrix} s \\ 0 \\ s \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 - x_3 = 0$ $\quad x_1 = s$

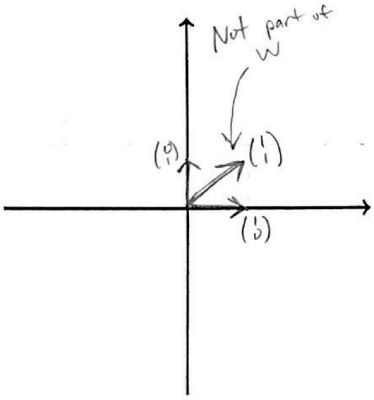
$x_2 = 0$ $\quad x_2 = 0$ $\quad s\begin{pmatrix}1\\0\\1\end{pmatrix}$

$x_3 = s$ $\quad x_3 = s$

3/3

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

4. **(4 pts)** In $\mathbb{R}^2$, consider the subset W consisting of the two coordinate axes (the solid lines in the picture below). Is W a linear subspace of $\mathbb{R}^2$? Motivate your answer.



No. W is not closed under addition. For example if we add vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ which are in the subspace, the resultant vector is $\begin{pmatrix}1\\1\end{pmatrix}$ which is not part of the subspace.

4/4

5. **(4 pts)** Consider the following vectors in $\mathbb{R}^4$: $\vec{v}_1 = \begin{pmatrix}1\\1\\7\\3\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}3\\3\\5\\4\end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$.

Is the vector $\begin{pmatrix}45\\18\\3\\1\end{pmatrix}$ in span$(\vec{v}_1, \vec{v}_2, \vec{v}_3)$? Motivate your answer.

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 45 \\ 1 & 3 & 0 & 18 \\ 7 & 5 & 1 & 3 \\ 3 & 4 & 0 & 1 \end{array}\right) \rightarrow \begin{array}{c} \text{I} \\ \text{II}-\text{I} \\ \text{III}-7\text{I} \\ \text{IV}-3\text{I} \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 0 & 45 \\ 0 & 0 & 0 & -27 \\ & & & \\ & & & \end{array}\right)$$

No. When solving this linear system we get an inconsistant result, $0 \neq -27$ or $0 \neq 1$. This means there are 0 lin. comb. of $\vec{v_1}, \vec{v_2}$, and $\vec{v_3}$ that make up $\begin{pmatrix}45\\18\\3\\1\end{pmatrix}$

4/4

6. Let A be the following matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}.$$

(a) **(10 pts)** Using row operations, find (if possible) the inverse matrix of A.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\substack{I \\ II-I \\ III-I}} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right) \xrightarrow{\substack{I-II \\ II \\ III-2II}} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \xrightarrow{\substack{I+III \\ II-2III \\ III}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

√10/10

(b) **(5 pts)** Find all the solutions of the following linear system:

$$A^{-1}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

$$I_3 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 1\begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} + 0\begin{pmatrix} -3 \\ 5 \\ -2 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \\ 3 \end{pmatrix}$$

√5/5

7. Let $T\colon \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation with associated matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 6 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) **(3 pts)** Write the image of T as the span of a set of vectors.

$$\operatorname{im}(T) = \operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 5 \\ 0 \end{pmatrix} \right)$$

✓ 3/3

(b) **(5 pts)** Find a basis for the image of T. Motivate why the set you exhibit is a basis (you may refer to a theorem, or argue directly).

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 6 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{matrix} \text{I} \\ \text{II} - 6\,\text{III} \\ \text{III} \\ \text{IV} \end{matrix} \begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -26 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \operatorname{rref}(A)$$

↑ ↑ ↑

✓ 5/5

The columns with leading 1's in rref(A) correspond to the basis vectors in A. All other columns are redundant.

Basis of $\operatorname{im}(T)$: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 1 \\ 0 \end{pmatrix}$