Math 33A-1 Midterm 1 Fall 2019

Name: ______________________ uid: ______________________

Section: __________ Signature: ______________________

Instructions:

- Unless otherwise stated, you need to justify your answer. Please show all of your work, as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown.
- You have 50 minutes to complete the exam.
- All answers should be completely simplified, unless otherwise stated.
- This is a closed book and closed notes test. You may **not** use a scientific calculator. No electronics are allowed on this exam. Make sure all cell phones are silenced, put away and out of sight. If you have a cell phone out at any point, for any reason, this will constitute a violation of test policy, and you may receive a zero on this exam.
- If asked, you must show us your **bruin card**.
- You may ask for scratch paper. You may use **no** other scratch paper. Please transfer all finished work onto the proper page in the test for us to grade there. We will **not** grade the work on the scratch page.
- Notice that the test is printed just front, so the space left for each question is sufficient, but possibly not necessary, to answer the questions. If you write on the back of a page, please indicate it.

STUDENT: PLEASE DO NOT WRITE BELOW THIS LINE. THIS TABLE IS TO BE USED FOR GRADING.

Problem	Points	Score
1	8	8
2	4	4
3	7	7
4	4	4
5	4	4
6	15	15
7	8	8
Total	50	50

1. **(8 pts)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformations determined by

$$\underbrace{T\begin{pmatrix}1\\0\end{pmatrix}}_{T(\vec{e_1})} = \begin{pmatrix}2\\1\end{pmatrix}, \quad \underbrace{T\begin{pmatrix}0\\1\end{pmatrix}}_{T(\vec{e_2})} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \underbrace{S\begin{pmatrix}1\\0\end{pmatrix}}_{S(\vec{e_1})} = \begin{pmatrix}0\\-1\end{pmatrix}, \quad \underbrace{S\begin{pmatrix}0\\1\end{pmatrix}}_{S(\vec{e_2})} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Find the matrix associated to the linear transformation $S \circ T\colon \mathbb{R}^2 \to \mathbb{R}^2$.

matrix A, $T = \begin{pmatrix}2 & 1\\1 & 0\end{pmatrix}$

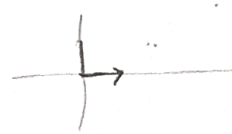
matrix B, $S = \begin{pmatrix}0 & 0\\-1 & 0\end{pmatrix}$

$S \circ T = BA = \begin{pmatrix}0 & 0\\-1 & 0\end{pmatrix}\begin{pmatrix}2 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}0+0 & 0+0\\-2+0 & -1+0\end{pmatrix} = \boxed{\begin{pmatrix}0 & 0\\-2 & -1\end{pmatrix}}$ ✓

8/8

2. **(4 pts)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function with

$$T\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}8\\5\end{pmatrix}, \quad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}, \quad T\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}5\\5\end{pmatrix}.$$



Can T be a linear transformation? If yes, provide an example. If not, provide an explanation.

A property of a linear transformation is that

$T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$.

If we set $\vec{v} = \begin{pmatrix}2\\1\end{pmatrix}$ and $\vec{w} = \begin{pmatrix}0\\1\end{pmatrix}$, we obtain

$T\left(\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\right) = T\begin{pmatrix}2\\2\end{pmatrix} = T\begin{pmatrix}2\\1\end{pmatrix} + T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}8\\5\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}9\\5\end{pmatrix}$

However, we are given that $T\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}5\\5\end{pmatrix} \neq \begin{pmatrix}9\\5\end{pmatrix}$, so T cannot be a linear transformation.

4/4

3. For each of the following cases, write a matrix satisfying the stated property, or state why it is impossible.

(a) (2 pts) A is a 5×6 matrix of rank 6.

Impossible; the rank of a matrix is bounded by n or m in a $n \times m$ matrix, whichever is smaller.

i.e., $\text{rank}(A) \leq n$ and $\text{rank}(A) \leq m$.

And since $n = 5$ and $m = 6$, the maximum rank for A is 5, making it impossible for the rank to be 6.

2/2

row x column

(b) (2 pts) A is a 3×2 matrix of rank 1. (1 pivot/leading 1).

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2/2

(c) (3 pts) A is a 3×3 matrix with $\ker(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$1x + 0y + 1z = 0$

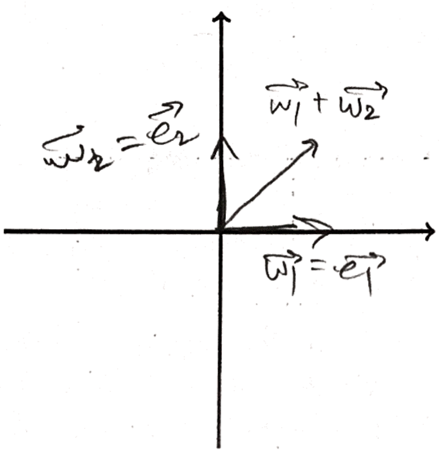
$(1 \quad 0 \quad -1)$
$(2 \quad 7 \quad -2)$
$(3 \quad 4 \quad -3)$

$$\rightsquigarrow \boxed{A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 7 & -2 \\ 3 & 4 & -3 \end{pmatrix}}$$

$$\begin{array}{c} \text{I} \\ \text{II}-2\text{I} \\ \text{III}-3\text{I} \end{array} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 7 & 0 \\ 0 & 4 & 0 \end{pmatrix} \begin{array}{c} \text{I} \\ \frac{1}{7}\text{II} \\ \text{III} \end{array} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} x_1 = x_3 \\ x_2 = 0 \\ x_3 = t \end{array} \quad \begin{pmatrix} t \\ 0 \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \checkmark$$

3/3

4. **(4 pts)** In $\mathbb{R}^2$, consider the subset W consisting of the two coordinate axes (the solid lines in the picture below). Is W a linear subspace of $\mathbb{R}^2$? Motivate your answer.



NO; by definition of a subspace, the subspace must be closed under addition. (i.e. if two vectors $\vec{w_1}$ and $\vec{w_2}$ are in subspace W, then $\vec{w_1}+\vec{w_2}$ is as well).

However, if we take $\vec{w_1}=\vec{e_1}$ and $\vec{w_2}=\vec{e_2}$, which are in subset W, then $\vec{w_1}+\vec{w_2}$ is no longer on the coordinate axes (see figure on the left). Thus W is not a linear subspace of $\mathbb{R}^2$.

4/4

5. **(4 pts)** Consider the following vectors in $\mathbb{R}^4$: $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 7 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 3 \\ 5 \\ 4 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

Is the vector $\begin{pmatrix} 45 \\ 18 \\ 3 \\ 1 \end{pmatrix}$ in $\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$? Motivate your answer.

$$c_1\begin{pmatrix} 1 \\ 1 \\ 7 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ 3 \\ 5 \\ 4 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 45 \\ 18 \\ 3 \\ 1 \end{pmatrix} \rightarrow \begin{matrix} c_1 + 3c_2 + 0c_3 = 45 \\ c_1 + 3c_2 + 0c_3 = 18 \\ (0 \quad 0 \quad 0 \mid 27) \end{matrix}$$

inconsistent system. No solution

The vector $\begin{pmatrix} 45 \\ 18 \\ 3 \\ 1 \end{pmatrix}$ is <u>not</u> in $\text{span}(\vec{v_1}, \vec{v_2}, \vec{v_3})$, since only $\vec{v_1}$ and $\vec{v_2}$ affect the first two entries in the column vector (x_1 and x_2 in $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$), and those first two entries of $\vec{v_1}$ and $\vec{v_2}$ always end up the same value through linear combinations. And since $45 \neq 18$, the vector $\begin{pmatrix} 45 \\ 18 \\ 3 \\ 1 \end{pmatrix}$ cannot be in the span

OR, equivalently, solving the associated linear system $\left(\begin{array}{ccc|c} 1 & 3 & 0 & 45 \\ 1 & 3 & 0 & 18 \\ 7 & 5 & 1 & 3 \\ 3 & 4 & 0 & 1 \end{array}\right)$ results in $\left(\begin{array}{ccc|c} 1 & 3 & 0 & 45 \\ 0 & 0 & 0 & -27 \\ 7 & 5 & 1 & 3 \\ 3 & 4 & 0 & 1 \end{array}\right)$, which is inconsistent.

4/4

6. Let A be the following matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}.$$

(a) **(10 pts)** Using row operations, find (if possible) the inverse matrix of A.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} \mathrm{I} \\ \mathrm{II}-\mathrm{I} \\ \mathrm{III}-\mathrm{I} \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right) \begin{array}{c} \mathrm{I}-\mathrm{II} \\ \mathrm{II} \\ \mathrm{III}-2\mathrm{II} \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\begin{array}{c} \mathrm{I}+\mathrm{III} \\ \mathrm{II}-2\mathrm{III} \\ \mathrm{III} \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \rightarrow \boxed{A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix}}$$

√10/10

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix} \begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} \cancel{3-3}+1 & -3+5-2 & 1-2+1 \\ 3-6+3 & -\cancel{3}+10-6 & 1-4+3 \\ 3-9+6 & -3+15-12 & 1-6+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

only one since invertible.

(b) **(5 pts)** Find all the solutions of the following linear system:

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}_{\vec{b}}. \qquad \vec{x} = A^{-1}(\vec{b})$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3+0+2 \\ -3+0-4 \\ 1+0+2 \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \\ 3 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \\ 3 \end{pmatrix}}$$

√5/5

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix} \begin{pmatrix} 5 \\ -7 \\ 3 \end{pmatrix} = \begin{pmatrix} 5-7+3 \\ 5-14+9 \\ 5-21+18 \end{pmatrix} = \begin{pmatrix} 8-7 \\ 14-14 \\ 23-21 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

7. Let $T\colon \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation with associated matrix

$$A = \begin{array}{c} \begin{array}{ccccc} \vec{v_1} & \vec{v_2} & \vec{v_3} & \vec{v_4} & \vec{v_5} \end{array} \\ \begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 6 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccccc} \uparrow & & \uparrow & \uparrow & \end{array} \end{array}.$$

$\vec{v_2} = 2\vec{v_1}$.
$\vec{v_5} = 5\vec{v_4} - 26\vec{v_3} + 3\vec{v_1}$] redundant.

(a) **(3 pts)** Write the image of T as the span of a set of vectors.

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 6 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{c} \mathrm{I} \\ \mathrm{II}-6\mathrm{III} \\ \mathrm{III} \\ \mathrm{IV} \end{array} \begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -26 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{im}(T) = \mathrm{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 1 \\ 0 \end{pmatrix} \right)$$

V 3/3

(b) **(5 pts)** Find a basis for the image of T. Motivate why the set you exhibit is a basis (you may refer to a theorem, or argue directly).

✓ 5/5

Basis for $\mathrm{im}(T)$: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 1 \\ 0 \end{pmatrix}$.

The vectors used in the span to describe the image forms the basis since they are linearly independent vectors that together span the im(T).