Math 33A-1 Final Summer 2020

For the instructions for this test, see the dedicated PDF.

1. **(6 pts)** Compute and completely factor the characteristic polynomial of the following matrix:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 5 & 1 & -4 \\ 0 & 1 & 1 \end{pmatrix}.$$

For credit, you **have to** factor the polynomial and show work for each step.

2. In the following, **use complex numbers if necessary**. For each of the following matrices:

   - compute the characteristic polynomial;
   - list all the eigenvalues (possibly complex) with their algebraic multiplicity;
   - for each eigenvalue, find a basis (possibly complex) of the corresponding eigenspace, and write the geometric multiplicity of the eigenvalue;
   - state whether the matrix is diagonalizable. If yes, write down an eigenbasis and the corresponding diagonal matrix. If not, motivate why.

   (a) **(5 pts)** $A = \begin{pmatrix} 3 & -4 \\ 1 & 3 \end{pmatrix}$

   (b) **(5 pts)** $B = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$

3. Consider the following matrix

$$A = \begin{pmatrix} 3 & 5 & 5 \\ 5 & 3 & 5 \\ 5 & 5 & 3 \end{pmatrix},$$

whose characteristic polynomial is $p_A(\lambda) = -(\lambda+2)^2(\lambda-13)$.

   (a) **(8 pts)** Find an orthonormal eigenbasis for the matrix A.

   (b) **(2 pts)** Find an orthogonal matrix S and a diagonal matrix D so that $D = S^{-1}AS$.

4. Let $q\colon \mathbb{R}^3 \to \mathbb{R}$ be the quadratic form given by $q(\vec{x}) = x_1^2 - 2x_2^2 + x_3^2 + 8x_1x_3$.

   (a) **(2 pts)** Find the symmetric matrix A associated to the quadratic form.

   (b) **(2 pts)** Is the quadratic form $q(\vec{x})$ positive definite, positive semi-definite, negative definite, negative semi-definite, or indefinite? Show your work.

5. Let B be the following matrix in reduced row-echelon form:

$$B = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) **(3 pts)** Let C be a matrix with $\text{rref}(C) = B$. Find a basis of $\ker(C)$.

(b) **(3 pts)** Find two matrices A_1 and A_2 so that $\text{rref}(A_1) = \text{rref}(A_2) = B$, and $\text{im}(A_1) \neq \text{im}(A_2)$.

(c) **(5 pts)** Find the matrix A with the following properties: $\text{rref}(A) = B$, $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalue 1, and $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalue 2. You can leave your answer as a product of matrices and inverse of matrices (you do not need to compute any inverse: if you need one, just leave it as $^{-1}$ of a suitable matrix). For credit, you **have to** motivate your choice. Note that checking that the matrix you exhibit works does not provide any evidence as for why you chose the matrix.

6. Consider the following matrix

$$A = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

(a) **(6 pts)** Determine the singular values of A.

(b) **(9 pts)** Find the singular value decomposition of A. Your answer has to consist of three matrices U, Σ, V satisfying the appropriate properties and multiplied together to retrieve A.

7. **(5 pts)** Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation with matrix

$$A = \begin{pmatrix} 2 & -2 & 1 & 0 \\ 2 & 2 & 0 & 1 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 2 & 2 \end{pmatrix}.$$

Let P be the 2-parallelepiped (i.e., parallelogram) determined by the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Determine the 2-volume (i.e., area) of $T(P)$.

8. For each of the following cases, provide an example satisfying the stated property, or state why it is impossible.

   (a) **(2 pts)** A 2×2 matrix with singular values 1 and 2 so that $||A\vec{e}_1|| = 5$.

   (b) **(2 pts)** A 3×3 matrix with 3 real distinct eigenvalues so that 0 is an eigenvalue of $A^2 + I_3$.