MATH 33A LECTURE 3
MIDTERM I

Please note: Show your work. Correct answers not accompanied by sufficent explanations will receive little or no credit (except on multiple-choice problems). Please call one of the proctors if you have any questions about a problem. No calculators, computers, PDAs, cell phones, or other devices will be permitted. *If you have a question about the grading or believe that a problem has been graded incorrectly, you must bring it to the attention of your professor within 2 weeks of the exam.*

#1	#2	#3	#4	#5	Total
10	10	10	10	7	47

Your section meets (circle):
Tuesday Thursday

TA name (circle):
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Jaehoon Lee
Austin Christian

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = [$

Student ID:________

Name:________

Signature:________

By signing above I certify that I am the person whose name and student ID appears on this page.

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Problem 1. (True/False, 1 pt each) Mark your answers by filling in the appropriate box next to each question (no explanations are necessary on this problem).

✓ (a) T F Let A and B be 5×5 matrices. If $\ker A = \operatorname{im} B$ then $AB = 0$. $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $B = \begin{bmatrix} l & m & n \\ o & p & q \\ r & s & t \end{bmatrix}$

rows & cols are orthogonal

✓ (b) T F If A is an invertible $n \times n$ matrix, then the rank of A is n.

✓ (c) T F There exists a 3×3 matrix A for which $\operatorname{im} A = \mathbb{R}^3$. I_3

X

✓ (d) T F If A is an $n \times m$ matrix and $\ker A = 0$, then $m \leq n$.

$n=4$, $m=3$ $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \\ j & k & l \end{bmatrix}$ $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ $n=2$, $m=3$

✓ (e) T F The matrix $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is in reduced row echelon form.

✓ (f) T F If $A \cdot A \cdot A \cdot A$ is the identity matrix, then A is invertible.

$A \cdot A \cdot A \cdot A = I_n$

✓ (g) T F If $\ker A = \operatorname{im} A$ for some square $n \times n$ matrix A, then n must be even.

if $A^4 = I_n$, $[(A \cdot A)(A \cdot A)] =$

$(A \cdot A)^{-1} = A \cdot A$

✓ (h) T F The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is invertible.

✓ (i) T F If T is a counterclockwise rotation by 2.016 radians in the plane, then $\ker T = (0)$.

✓ (j) T F The set $\{(x, y) : xy = 1\}$ is not a subspace of $\mathbb{R}^2$.

doesn't contain 0 ?

if $(A \cdot A)^{-1} \neq A^{-1}A^{-1}$

??? All true ???

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$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$AB = I_n$

$(A \cdot A)(A \cdot A) = I_n$

$A \cdot A = (A \cdot A)^{-1}$

Problem 2. (10 pts) Let $A = \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \\ \end{array}$ $\begin{bmatrix} 0 & 1 & 4 & 7 & 1 \\ 1 & 2 & 5 & 8 & 1 \\ 1 & 3 & 6 & 9 & 2 \end{bmatrix}$.

$$\text{rref}\left(\begin{bmatrix} 1 & 2 & 5 & 8 & 1 \\ 1 & 3 & 6 & 9 & 2 \\ 0 & 1 & 4 & 7 & 1 \end{bmatrix}\right)$$

(a) Are the columns of A linearly independent?

No

(last 2 columns are not)

$$\rightarrow \begin{bmatrix} 1 & 2 & 5 & 8 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 6 & -1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$$

$$\begin{array}{ccccc} & & & 3 & 6 \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$$

(b) Find a basis for the image of A.

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

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(c) Find a basis for the kernel of A.

$$V_4 = 2V_3 - V_2 \Rightarrow V_4 - 2V_3 + V_2 = 0$$

$$V_5 = V_2 - V_1 \Rightarrow V_5 - V_2 + V_1 = 0$$

$$\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Problem 3. (10 pts) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation whose matrix is given by:

$$\begin{array}{cc} V_1 & V_2 \end{array}$$
$$A = \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}. \qquad \begin{matrix} a & b \\ b & 3a \end{matrix} \qquad \begin{matrix} w_1 = \frac{1}{2} \\ w_2 = \frac{\sqrt{3}}{2} \end{matrix}$$

$\sqrt{3}/4 \quad 3/4$

(a) Find $\ker A$.

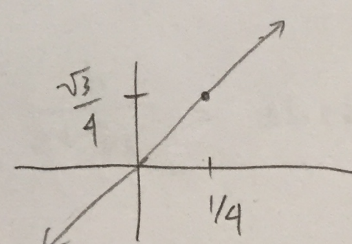
$$\mathrm{Ker}(A) = \left[\begin{array}{cc|c} 1/4 & \sqrt{3}/4 & 0 \\ \sqrt{3}/4 & 3/4 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1/4 & \sqrt{3}/4 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \frac{1}{4}x_1 = -\frac{\sqrt{3}}{4}t$$

$$x_1 = -\sqrt{3}\,t$$
$$x_2 = t$$

$$t\begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} \Rightarrow \mathrm{span}\begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} \checkmark$$

(b) Find $\operatorname{im} A$.

$$\mathrm{span}\begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix} \checkmark \qquad V_2 = \sqrt{3}\,V_1$$



(c) Describe the transformation S geometrically.

$$\begin{bmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1+\sqrt{3} \\ 3+\sqrt{3} \end{bmatrix}$$

S is a projection onto the line $y = \sqrt{3}\,x$. ✓

$$\frac{\frac{\sqrt{3}}{4}}{1/4} = \frac{\sqrt{3}}{\cancel{4}}$$

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Problem 4. (10 pts) Find ~~an~~ only one needed invertible 2×2 matrix A so that $A \cdot A = A^{-1}$ (in other words, $A^2 = A^{-1}$). Hint: try to find the corresponding transformation first.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad A \cdot A = A^{-1}$

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & cb+d^2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$

$\frac{1}{9-0}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$$\begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & cb+d^2 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\frac{1}{1-0}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

① $\frac{d}{ad-bc} = a^2+bc$ ② $-\frac{b}{ad-bc} = ab+bd$

③ $-\frac{c}{ad-bc} = ac+dc$ ④ $\frac{a}{ad-bc} = cb+d^2$

$d = (a^2+bc)(ad-bc)$ $\quad b = -(ab+bd)(ad-bc)$

$c = -(ad-bc)(ac+dc)$ $\quad a = (ad-bc)(cb+d^2)$

$d = a^3 d - a^2bc + abcd - b^2c^2$ $\quad b =$

$d(1-abc) = a^3d - a^2bc - b^2c^2$

$d = \frac{a^3d - a^2bc - b^2c^2}{1-abc}$

$\boxed{A = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$ ✓

$y = -2z$

$x - 2z + z = 0$

$x - z = 0$

$x - 4z + 3z = 0$

$x - z = 0$ 6

Problem 5. (10 pts) Let P be the plane $x + y + z = 0$ and Q be the plane $x + 2y + 3z = 0$. Their intersection is a subspace which is a line. Find a basis for that subspace.

$$\begin{aligned} x + y + z &= 0 \\ - \; x + 2y + 3z &= 0 \\ \hline -y - 2z &= 0 \\ y + 2z &= 0 \end{aligned}$$

$\leftarrow$ intersection.

$\cancel{x} + y + z = \cancel{x} + 2y + 3z$

$-2z = y$

$x = z$

Basis of intersection is any nonzero vector that is on the line, so one of them is

$$\begin{bmatrix} \cancel{0}\ 1 \\ 2 \\ -1 \end{bmatrix}.$$

since $2 + 2(-1) = 0$

(thus, the entire subspace is span $\left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\right)$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$x = z$ $\quad z = z$

$y = -2z$

$z\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$