Math 33A

Midterm I solutions

Problem 1. (True/False, 1 pt each) Mark your answers by filling in the appropriate box next to each question (no explanations are necessary on this problem).

(a $\boxed{T}\boxed{F}$) Let A and B be 5×5 matrices. If $\ker A = \operatorname{im} B$ then $AB = 0$.

(b $\boxed{T}\boxed{F}$) If A is an invertible $n \times n$ matrix, then the rank of A is n.

(c $\boxed{T}\boxed{F}$) There exists a 3×3 matrix A for which $\operatorname{im} A = \mathbb{R}^3$.

(d $\boxed{T}\boxed{F}$) If A is an $n \times m$ matrix and $\ker A = 0$, then $m \leq n$.

(e $\boxed{T}\boxed{F}$) The matrix $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is in reduced row echelon form.

(f $\boxed{T}\boxed{F}$) If $A \cdot A \cdot A \cdot A$ is the identity matrix, then A is invertible.

(g $\boxed{T}\boxed{F}$) If $\ker A = \operatorname{im} A$ for some square $n \times n$ matrix A, then n must be even.

(h $\boxed{T}\boxed{F}$) The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is invertible.

(i $\boxed{T}\boxed{F}$) If T is a counterclockwise rotation by 2.016 radians in the plane, then $\ker T = (0)$.

(j $\boxed{T}\boxed{F}$) The set $\{(x, y) : xy = 1\}$ is not a subspace of $\mathbb{R}^2$.

all true

Problem 2. (10 pts) Let $A = \begin{bmatrix} 0 & 1 & 4 & 7 & 1 \\ 1 & 2 & 5 & 8 & 1 \\ 1 & 3 & 6 & 9 & 2 \end{bmatrix}$.

(a) Are the columns of A linearly independent?

No eg $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

OR rank $A \leq 3$ because A is 3×5

so can have at most 3 lin. indep. columns.

(b) Find a basis for the image of A.

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix} \Rightarrow \text{col's } 1,2,3 \text{ form a basis: } \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right)$$

$$A \sim \begin{bmatrix} 1 & 3 & 6 & 9 & 2 \\ 1 & 2 & 5 & 8 & 1 \\ 0 & 1 & 4 & 7 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 6 & 9 & 2 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 4 & 7 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 6 & 9 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 6 & -1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$$

(c) Find a basis for the kernel of A.

$$A \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = 0 \iff \text{rref}(A) \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = 0 \Rightarrow$$

$x_1 = +t$
$x_2 = s - t$
$x_3 = -2s$
$x_4 = s$
$x_5 = t$

$$\Rightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \text{Basis} = \left(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right)$$

Problem 3. (10 pts) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation whose matrix is given by:

$$A = \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}.$$

(a) Find $\ker A$.

$$A \sim \begin{bmatrix} 1 & \sqrt{3} \\ 1 & \sqrt{3} \end{bmatrix} \sim \begin{bmatrix} 1 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \iff \begin{bmatrix} 1 & \sqrt{3} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} x_1 = -\sqrt{3} \cdot s \\ x_2 = s \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left\{ s \cdot \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} \right\}$$

(b) Find $\operatorname{im} A$.

$$\operatorname{im} A = \operatorname{span} \begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \right)$$

(c) Describe the transformation S geometrically.

Since $\ker S$ is a line $\perp$ $\operatorname{im} S$ and $v = S(v)$ iff v is in $\operatorname{im} S$, it follows that S = orthogonal project onto span $\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$.

Problem 4. (10 pts) Find an invertible 2×2 matrix A so that $A \cdot A = A^{-1}$ (in other words, $A^2 = A^{-1}$). Hint: try to find the corresponding transformation first.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A = I$ works (since $I \cdot I = I = I^{-1}$)

as does rotation by 120°.

Problem 5. (10 pts) Let P be the plane $x+y+z=0$ and Q be the plane $x+2y+3z=0$. Their intersection is a subspace which is a line. Find a basis for that subspace.

The intersection is the solution set to

$$\begin{cases} x+y+z=0 \\ x+2y+3z=0 \end{cases} \quad \text{i.e.} \quad \ker\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

$$\ker = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ s.t. } \begin{matrix} x = t \\ y = -2t \\ z = t \end{matrix} \right\} =$$

$$= \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t \right\}$$

$$\text{basis} = \left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right).$$