Math 33A **Name:** ____________________

ID Number ____________________

Winter 2021
Midterm 1
02/24/21

This exam contains 10 pages (including this cover page) and 4 problems.

You are required to show your work on each problem on this exam. Please take care to fully justify your answers, as an answer without the relevant work will receive very little credit. The following rules apply:

- Open book and open notes, but you may not use any other outside resources (no calculator allowed).
- You don't need to print this out and write directly on the exam, you may submit separate sheets of paper. If you print out and write on the exam, you may use the provided blank pages for your work. In either case, make sure you organize your work, in a reasonably neat and coherent way. Work scattered all over the page without a clear ordering will receive very little credit.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. Let M be the $5x3$ matrix with linearly independent columns

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

   (a) (5 points) Let $\mathcal{A}$ be the basis of $\mathrm{im}(M)$ consisting of the columns of M. Find the orthonormal basis $\mathcal{B}$ obtained by applying Gram-Schmidt to $\mathcal{A}$.

   (b) (5 points) Use the previous part to find the QR factorization of M, being careful to justify your work.

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2. (a) (2 points) Explain why a least squares solution to a linear system $A\vec{x} = \vec{b}$ always exists, even if the system is inconsistent.

   (b) (5 points) Use least squares to fit a cubic function to the three data points $(2, 3)$, $(0, -1)$, $(3, 4)$.

(c) (3 points) Find the $4x4$ matrix corresponding to orthogonal projection onto the hyperplane $2x + 3y - 4z + w = 0$ in $\mathbb{R}^4$. Be sure to show all your steps.

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3. (a) (2 points) Compute the determinant of the following 6x6 matrix using patterns, being careful to show your steps:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(b) (3 points) Compute the determinant of the following 5x5 matrix using patterns, being careful to show your steps:

$$\begin{pmatrix} 1 & 0 & 3 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 1 \end{pmatrix}$$

(c) (5 points) Find the determinant of the $100x100$ matrix corresponding to a linear transformation T that acts the standard basis vectors $\vec{e_1}, \vec{e_2}, \ldots, \vec{e_{100}}$ in the following fashion.

$$T(\vec{e_1}) = \vec{e_2},$$

$$T(\vec{e_2}) = \vec{e_3},$$

$$\vdots$$

$$T(\vec{e_{99}}) = \vec{e_{100}},$$

$$T(\vec{e_{100}}) = \vec{e_1}.$$

That is, T takes each of the first 99 standard basis vectors to the subsequent standard basis vector, and T takes the last standard basis vector to the first standard basis vector. Be sure to justify all your reasoning.

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4. Let A be a $6x3$ matrix and B be a $3x4$ matrix. In this problem, most of the points will be awarded for correct justification.

   (a) (3 points) What are the possible dimensions of the kernel of AB? Justify your answer.

   (b) (2 points) What are the possible dimensions of the image of AB? Justify your answer.

   (c) (3 points) If the dimension of the kernel of B is 2, what is the dimension of the image of $B^T B$? Here B^T denotes the transpose of B. Justify your answer.

   (d) (2 points) If the kernel of B is a hyperplane in $\mathbb{R}^4$, what is the dimension of the image of BB^T? Justify.

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