1. Consider the matrix 4x3 matrix A with linearly independent columns

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

(a) (5 points) Let $\mathcal{A}$ be the basis of im(A) consisting of the columns of A. Find the orthonormal basis $\mathcal{B}$ obtained by applying Gram-Schmidt to $\mathcal{A}$.

(b) (5 points) Use the previous part to write down the QR factorization of A, being careful to justify your work.

a) $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$

$\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad \text{proj}_{\vec{u}_1}\vec{v}_2 = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}(2)\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\vec{v}_2^{\perp} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$\vec{v}_3^{\parallel} = \text{proj}_{\vec{u}_1}\vec{v}_3 + \text{proj}_{\vec{u}_2}\vec{v}_3$ $\quad \text{proj}_{\vec{u}_1}\vec{v}_3 = \frac{1}{2}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\text{proj}_{\vec{u}_2}\vec{v}_3 = \left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad = \frac{1}{2}(2)\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\vec{v}_3^{\parallel} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad \vec{v}_3^{\perp} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ $\quad \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$$\boxed{B = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}}$$

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b) $M = QR$

$$\boxed{Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}_Q = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}_Q = \begin{bmatrix} \sqrt{2} \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}_Q = \begin{bmatrix} \sqrt{2} \\ 1 \\ 2 \end{bmatrix}$$

$$R = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}_Q & \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}_Q & \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}_Q \end{bmatrix}$$

$$\boxed{R = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}}$$

1 Question 1 **10 / 10**

✓ **- 0 pts Correct answer: $$Q** = **\begin{bmatrix}
1/\sqrt{2} & 0 & 0\\
0 & 1 & 0\\
0 & 0 & 1\\
1/\sqrt{2} & 0 & 0
\end{bmatrix}$$, $$R** = **\begin{bmatrix}
\sqrt{2} & \sqrt{2} & \sqrt{2}\\
0 & 1 & 1\\
0 & 0 & 2
\end{bmatrix}$$**

- **- 2 pts** Error in computing the $$\vec{u}$$'s
- **- 2 pts** Error in computing $$R$$
- **- 1 pts** Normalization error
- **- 2 pts** Multiple computational errors
- **- 3 pts** Missing calculation of $$R$$
- **- 3 pts** Computed $$R^{-1}$$ instead of $$R$$
- **- 1 pts** Minor calculation error
- **- 3 pts** Lacking justification for computing $$R$$

2. (a) (6 points) Suppose that A is a $3x3$ matrix given by

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Let $\mathcal{B}$ be the basis of $\mathbb{R}^3$ given by

$$\mathcal{B} = \{(1,0,1),(1,0,0),(0,1,0)\}.$$

Find the $\mathcal{B}$-matrix of A; that is find the matrix B satisfying

$$[T(\vec{v})]_{\mathcal{B}} = B[\vec{v}]_{\mathcal{B}}$$

for all $\vec{v}$ in $\mathbb{R}^3$. Show all your steps.

(b) (4 points) (Unrelated to part (a)). Suppose that A is an mxn matrix corresponding to a linear transformation that is injective, and suppose $\vec{b}$ is a vector contained in $(im(A))^{\perp}$. Show that the least squares solution $\vec{x}^*$ to the system $A\vec{x} = \vec{b}$ is $\vec{x}^* = \vec{0}$.

a) $S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R1-R3} \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \qquad S^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$B = S^{-1}AS$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

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b) If $T(x)$ is injective, that means that no two vectors can be mapped to the same vector in the image of A.

For least squares, x^* is the vector such that $Ax^* = \text{proj}_{\text{im}(A)}\vec{b}$. Since $\vec{b}$ is in $\text{im}(A)^\perp$, it's projection onto $\text{im}(A)$ is $\vec{0}$. Taking the previous statement about the implications of injective, this means that for $Ax^* = \text{proj}_{\text{im}(A)}\vec{b} = \vec{0}$, there is only one vector x^* that will be mapped to $\vec{0}$. Given that $T(x)$ is a linear transformation, the $\vec{0}$ will always map to $\vec{0}$. Since $T(x)$ is injective $x^* = \vec{0}$

2 Question 2 **10 / 10**

✓ **+ 6 pts a) full credit: compute B={3, 1, 0; 0, 1, 0; 0, 0, 1], and as work compute \[T(v)]_{\\beta} for each basis vector v. Alternatively compute the change of basis matrix S and B=S^{-1}AS**

+ 4 pts a) partial credit: Attempted using change of basis but computed using the wrong change of basis matrix, or alternatively wrote out the formula for B in terms of the \[T(v_i)]_{\\beta} as column vectors but did not compute the \[T(v_i)]_{\\beta} correctly

+ 4 pts a) Partial credit: Computed T(basis vectors) and used them as columns of matrix instead of computing \[T(v)]_{\\beta}

+ 4 pts a) partial credit: other category - overall correct approach used but several steps where execution is incorrect

+ 3 pts a) partial credit: correct answer but no justifying work]

✓ **+ 4 pts b) full credit: either use Ax*=proj_{im(A)}b or formula for least squares solution x*. In the former case may notice that b in im(A perp) implies the right hand side of the above is the 0 vector, and then A injective implies x*=0. If you are using the formula, need to note that A^Tb=0.**

+ 2 pts b) partial credit: Proof has some correct statements in the right direction and some incorrect ones, or alternatively proof is missing steps

+ 0 pts 0 points

3. (a) (5 points) Let A be the 100x100 matrix that has all -1's below the diagonal, is 1 for every entry in the first row, and is 0 for all diagonal entries except for the first, depicted below.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \cdots & 1 & 1 \\ -1 & 0 & 0 & 0 \cdots & 0 & 0 \\ -1 & -1 & 0 & 0 \cdots & 0 & 0 \\ -1 & -1 & -1 & 0 \cdots & 0 & 0 \\ \vdots & & & & \vdots & \vdots \\ -1 & -1 & -1 & -1 \cdots & 0 & 0 \\ -1 & -1 & -1 & -1 \cdots & -1 & 0 \end{pmatrix}$$

Find the determinant of the matrix A, being sure to justify your work. (Hint: There is a trick that makes this much easier. Are there certain things we can do to a matrix that preserve determinants?)

(b) (5 points) Let A be an nxn matrix such that $A^m = 0$ for some positive integer m. (Here the 0 denotes the zero matrix, i.e the nxn matrix that has 0 in every entry, and the notation A^m means m copies of A multiplied together.) Prove that A is noninvertible.

a) If we add the first row to all of the following rows, we form an upper triangular matrix of all 1's. Since row addition does not affect determinant, we can find det(A) by taking the product of all the values along the diagonal so that $\boxed{\det(A) = 1}$

b) By determinant properties, we know $\det(AB) = \det(A)\det(B)$
We can rewrite A^m as the product of m A matrices
$A^m = AAAA \ldots \underset{m^{th} \text{ matrix}}{A}$
Using the previous property of determinants we know
$\det(A^m) = \det(A)\det(A)\det(A)\ldots\det(A) = (\det(A))^m$
As a result we can write $(\det(A))^m = \det(0)$
where $\det(0) = 0$ so $\det(A) = 0$. Since $\det(A) = 0$,
A is noninvertible.

3 Question 3 **10 / 10**

- ✓ + **2 pts** **(a) The determinant is 1**
- ✓ + **3 pts** **(a) Full justification**
- + **2 pts** (a) (partial) Some work towards a right answer
- + **1 pts** (a) (partial) Some justification
- ✓ + **5 pts** **(b) Fully correct proof**
- + **2 pts** (b) (partial) Some work towards a solution
- + **1 pts** (b) (partial) $$\det A^m=0$$
- + **1 pts** (b) (partial) $$\det A^m = (\det A)^m$$
- + **1 pts** (b) (partial) Therefore, $$\det A = 0$$
- + **0 pts** Blank or incorrect

4. Let A be the matrix

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 4 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

(a) (3 points) Find the eigenvalues of A.

(b) (3 points) Find the corresponding eigenspaces of A. List the algebraic and geometric multiplicities of each eigenvalue.

(c) (4 points) Explain why part b) tells you that A is diagonalizable. Diagonalize A by finding S, B so that $A = SBS^{-1}$ and B is diagonal.

a) $\det(A - \lambda I_n) = 0$

$$\det \begin{bmatrix} -\lambda & 2 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 2 & -\lambda \end{bmatrix} = 0$$

$$-\lambda((4-\lambda)(-\lambda) - 2) - 2(-\lambda - 1) + 1(2 - (4-\lambda)) = 0$$

$$-\lambda(-4\lambda + \lambda^2 - 2) + 2\lambda + \cancel{2} + \cancel{2} - \cancel{4} + \lambda = 0$$

$$4\lambda^2 - \lambda^3 + 2\lambda + 2\lambda + \lambda = 0$$

$$-\lambda^3 + 4\lambda^2 + 5\lambda = 0$$

$$\lambda^3 - 4\lambda^2 - 5\lambda = 0$$

$$\lambda(\lambda - 5)(\lambda + 1) = 0$$

$$\boxed{\begin{aligned} \lambda &= 0 \\ \lambda &= 5 \\ \lambda &= -1 \end{aligned}}$$

b) $E_0 = \ker \begin{bmatrix} 0 & 2 & 1 \\ 1 & 4 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 4 & 1 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow{R3-R1} \begin{bmatrix} 0 & 2 & 1 \\ 1 & 4 & 1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{R2+R3}$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & -1 \end{bmatrix} \div 2 \rightarrow \begin{bmatrix} 0 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{R2-R1} \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{R3-R2} \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - x_3 = 0$ $\quad x_1 = x_3$ $\quad x_3 = 2$

$2x_2 + x_3 = 0$ $\quad x_2 = \frac{-x_3}{2}$

$\lambda = 0$ $\quad E_0 = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \right\}$

algebraic multiplicity = 1

geometric multiplicity = 1

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$$E_5 = \ker \begin{bmatrix} -5 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -5 \end{bmatrix} \xrightarrow[R3+2R2]{R1+2R2} \begin{bmatrix} -3 & 0 & 3 \\ 1 & -1 & 1 \\ 3 & 0 & -3 \end{bmatrix} \xrightarrow{R3+R1} \begin{bmatrix} -3 & 0 & 3 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\div -3} \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R2-R1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{ll} x_1 - x_3 = 0 & x_1 = x_3 \\ -x_2 + 2x_3 = 0 & x_2 = 2x_3 \end{array}$$

$x_3 = 1$

$$\boxed{\lambda = 5 \quad E_5 = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\} \quad \begin{array}{l} \text{algebraic multiplicity} = 1 \\ \text{geometric multiplicity} = 1 \end{array}}$$

$$E_{-1} = \ker \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[R2-R1]{R3-R1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\div 3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R1-2R2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{lll} x_1 + x_3 = 0 & x_1 = -x_3 & x_3 = 1 \\ x_2 = 0 & & \end{array}$$

$$\boxed{\lambda = -1 \quad E_{-1} = \text{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\} \quad \begin{array}{l} \text{algebraic multiplicity} = 1 \\ \text{geometric multiplicity} = 1 \end{array}}$$

c) We know that A is diagonizable because, for each eigenvalue, the geometric multiplicity and the algebraic multiplicity are equal.

$$\boxed{S = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{bmatrix}}$$

4 Question 4 **10 / 10**

✓ **+ 10 pts** **Correct**

- **- 2 pts** Error in calculating characteristic polynomial
- **- 1 pts** Diagonalizability justification wrong
- **- 1 pts** B’s columns in the wrong order
- **- 1 pts** Small algebraic error
- **- 1 pts** Eigenspace mistake (this and the next two items are deducted in proportion to the number and gravity of mistakes in calculating the eigenspaces in part b)
- **- 1 pts** Eigenspace mistake
- **- 1 pts** Eigenspace mistake

1 Not exactly; algebraic multiplicities only sum to the dimension if you use complex numbers