1. Consider the matrix 4x3 matrix A with linearly independent columns

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

(a) (5 points) Let $\mathcal{A}$ be the basis of im(A) consisting of the columns of A. Find the orthonormal basis $\mathcal{B}$ obtained by applying Gram-Schmidt to $\mathcal{A}$.

(b) (5 points) Use the previous part to write down the QR factorization of A, being careful to justify your work.

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2. (a) (6 points) Suppose that $A$ is a $3x3$ matrix given by

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Let $\mathcal{B}$ be the basis of $\mathbb{R}^3$ given by

$$\mathcal{B} = \{(1,0,1), (1,0,0), (0,1,0)\}.$$

Find the $\mathcal{B}$-matrix of $A$; that is find the matrix $B$ satisfying

$$[T(\vec{v})]_{\mathcal{B}} = B[\vec{v}]_{\mathcal{B}}$$

for all $\vec{v}$ in $\mathbb{R}^3$. Show all your steps.

(b) (4 points) (Unrelated to part (a)). Suppose that $A$ is an $mxn$ matrix corresponding to a linear transformation that is injective, and suppose $\vec{b}$ is a vector contained in $(im(A))^{\perp}$. Show that the least squares solution $\vec{x}^*$ to the system $A\vec{x} = \vec{b}$ is $\vec{x}^* = \vec{0}$.

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3. (a) (5 points) Let $A$ be the 100x100 matrix that has all $-1$'s below the diagonal, is 1 for every entry in the first row, and is 0 for all diagonal entries except for the first, depicted below.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \cdots & 1 & 1 \\ -1 & 0 & 0 & 0 \cdots & 0 & 0 \\ -1 & -1 & 0 & 0 \cdots & 0 & 0 \\ -1 & -1 & -1 & 0 \cdots & 0 & 0 \\ \vdots & & & & \vdots & \vdots \\ -1 & -1 & -1 & -1 \cdots & 0 & 0 \\ -1 & -1 & -1 & -1 \cdots & -1 & 0 \end{pmatrix}$$

Find the determinant of the matrix A, being sure to justify your work. (Hint: There is a trick that makes this much easier. Are there certain things we can do to a matrix that preserve determinants?)

(b) (5 points) Let $A$ be an nxn matrix such that $A^m = 0$ for some positive integer $m$. (Here the 0 denotes the zero matrix, i.e the nxn matrix that has 0 in every entry, and the notation $A^m$ means $m$ copies of $A$ multiplied together.) Prove that $A$ is noninvertible.

4. Let $A$ be the matrix

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 4 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

   (a) (3 points) Find the eigenvalues of $A$.

   (b) (3 points) Find the corresponding eigenspaces of $A$. List the algebraic and geometric multiplicities of each eigenvalue.

   (c) (4 points) Explain why part b) tells you that $A$ is diagonalizable. Diagonalize $A$ by finding $S, B$ so that $A = SBS^{-1}$ and $B$ is diagonal.

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