# 20S-MATH33A-2 Midterm 1

FRANK ZHENG

TOTAL POINTS

**38 / 40**

QUESTION 1

## 1 Question 1 8 / 10

+ **10 pts** Correct
+ **4 pts** (a) correct
- ✓ + **3 pts** **(a) one mistake**
+ **2 pts** (a) two or more mistakes, or a more serious mistake.
- ✓ + **1 pts** **(b) not injective**
- ✓ + **1 pts** **(b) surjective**
- ✓ + **1 pts** **(c) not invertible**
- **1 pts** (b) did not explain fully

it more difficult to get this deduction if full points were not otherwise awarded for (b)

+ **3 pts** (c) correct
- ✓ + **2 pts** **(c) one mistake**
+ **1 pts** (c) two mistakes or more serious mistakes.

1 Missing a column

2 No, the kernel is a subset of R^4; should be (t, -t, 0, t)

QUESTION 2

## 2 Question 2 10 / 10

- ✓ + **1 pts** **(a) Set up equation $$A\vec{x} = \vec{0}$$ or set up row reduction.**
- ✓ + **1 pts** **(a) Row reduced $$A$$ correctly, or solved system correctly**
- ✓ + **1 pts** **(a) Given answer spans the kernel**
- ✓ + **1 pts** **(a) Given answer is linearly indepedent in $$\mathbb{R}^3$$**
- ✓ + **1 pts** **(b) Said image is span of columns (or other correct formulation)**
- ✓ + **1 pts** **(b) Given answer spans image**
- ✓ + **1 pts** **(b) Given answer is linearly independent in $$\mathbb{R}^3$$**
- ✓ + **1 pts** **(c) Kernel is a line in $$\mathbb{R}^3$$**
- ✓ + **1 pts** **(c) Image is a plane in $$\mathbb{R}^3$$**
- ✓ + **1 pts** **(c) Kernel and image pass through the origin, or an explicit description of which line and plane correspond to the kernel and image.**
+ **0 pts** Blank or incorrect

QUESTION 3

## 3 Question 3 10 / 10

- ✓ + **2 pts** **a) full credit: Starting approach of using linearity to deduce T(e_3), or finding T^{-1} to deduce T**
- ✓ + **3 pts** **a) full credit: Putting T(e_j) into columns of matrix, or using Gaussian elimination to find T from T^{-1}**
+ **1 pts** a) partial credit: Some kind of approach given that leads to the right answer, but not clearly explained; or, linearity appears to be used, but it is not completely clear
+ **1 pts** a) partial credit: formula for A in terms of T(e_j) correct, but incorrectly executed
+ **1 pts** a) partial credit, first two columns of matrix correct, but no clearly written correct reasoning
+ **1 pts** a) partial credit: Some kind of attempt that vaguely makes sense, but is still incorrect and doesn't lead anywhere near the correct answer
- **1 pts** computation error
- **1 pts** computation error
- ✓ + **3 pts** **b) full credit: correct answer**
- ✓ + **2 pts** **b) full credit: justifying work**
+ **1 pts** b) partial credit: some but not sufficient justification
+ **1 pts** b) partial credit: partially correct work that leads to the wrong answer

QUESTION 4

## 4 Question 4 10 / 10

✓ + 3 pts (a):

(1): show $$V_{\vec{b}}$$ does not contain $$\vec{0}$$.
Proof: $$A\vec{0} = \vec{0}$$, but $$\vec{0} \neq \vec{b}$$ by assumption; so $$\vec{0}\notin V_{\vec{b}}$$ by definition

(2): show $$V_{\vec{b}}$$ is not closed under vector addition.
Proof: pick $$\vec{v} \in V_{\vec{b}}$$; then $$A(\vec{v}) = \vec{b}$$; but $$A(\vec{v}+\vec{v}) =\vec{b}+\vec{b}= 2\vec{b}$$, which is not equal to $$\vec{b}$$ since $$\vec{b}\neq \vec{0}$$ (by assumption); so $$\vec{v}+\vec{v} \notin V_{\vec{b}}$$

(3): show $$V_{\vec{b}}$$ is not closed under scalar multiplication.
Proof: pick $$\vec{v} \in V_{\vec{b}}$$; then $$A(\vec{v}) = \vec{b}$$; but $$A(-1\vec{v}) = -\vec{b}$$, which is not equal to $$\vec{b}$$ since $$\vec{b}\neq \vec{0}$$ (by assumption); so $$-1\vec{v} \notin V_{\vec{b}}$$

\[ We assume $$V_{\vec{b}}$$ is nonempty for (2) and (3); however, if the set $$V_{\vec{b}}$$ is empty, then it is not a subspace in this case either ]

✓ + 1 pts (b): show $$(\vec{x}+\mathrm{ker}A) \subseteq V_{\vec{b}}$$, assuming $$\vec{x} \in V_{\vec{b}}$$.
Proof: let $$\vec{y} \in \mathrm{ker}A$$ be given.
Then $$A\vec{y}=\vec{0}$$, and
$$A\vec{x}=\vec{b}$$ already (by assumption);
so $$A(\vec{x}+\vec{y}) = A\vec{x}+A\vec{y} = \vec{0}+\vec{b}=\vec{b}$$,
so $$\vec{x}+\vec{y} \in V_{\vec{b}}$$.

✓ + 2 pts (b): show $$V_{\vec{b}} \subseteq(\vec{x}+\mathrm{ker}A)$$, assuming $$\vec{x} \in V_{\vec{b}}$$.
Proof: let $$\vec{u} \in V_{\vec{b}} $$ be given. Then $$A\vec{u}=\vec{b}$$, and already $$A\vec{x} = \vec{b}$$ (by assumption).
Define $$\vec{y} = \vec{u}-\vec{x}$$; now $$A\vec{y} = A(\vec{u}-\vec{x})=A\vec{u}-A\vec{x}=\vec{b}-\vec{b}=\vec{0}$$; therefore $$\vec{y} \in \mathrm{ker}A$$.
So now $$\vec{u} = \vec{x}+(\vec{u}-\vec{x})= \vec{x}+\vec{y}$$, and $$\vec{y} \in \mathrm{ker}A$$; therefore $$\vec{u} \in \vec{x}+\mathrm{ker}A$$.

✓ + 2 pts (c): basis for $$\mathrm{im}A$$ has $$m-1$$ vectors (by e.g. Rank Theorem; no justification needed in answer for this part)

✓ + 1 pts (d): $$A$$ never injective (because kernel is nontrivial (since we are told the kernel is a line and therefore has dimension $$1>0$$));
so also never invertible (because invertible is equivalent to being injective and surjective simultaneously)
(no justification needed for this part)

✓ + 1 pts (d): $$A$$ sometimes surjective. (no justification needed for this part)
In fact, in the case of this problem (if $$\mathrm{ker}A$$ is a line and $$A$$ is an $$n\times m$$ matrix), $$A$$ will be surjective iff $$n=m-1$$.

1. Consider the following system of linear equations.

$$x_1 + x_2 + x_3 = 1$$

$$x_1 + x_2 + 2x_3 = 2$$

$$x_2 + x_4 = 2$$

   (a) (4 points) Rewrite the system as a matrix equation $A\vec{x} = \vec{b}$ and convert the augmented matrix $[A\,|\,\vec{b}]$ to reduced row echelon form.

   (b) (3 points) Is the linear transformation T corresponding to A injective? Surjective? Invertible?

   (c) (3 points) Write down the solution set to the linear system.

a.

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \vec{x} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{\vec{b}}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - R_2 - R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

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b. It is not injective because it maps from $\mathbb{R}^4$ (a higher dimension Euclidean space) to $\mathbb{R}^3$ (a lower dimension Euclidean space). More rigorously:

$$\ker(A) = \ker(\mathrm{rref}(A)) = \ker\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\rightarrow x_1 = x_4,\ x_2 = -x_4,\ x_3 = 0$

$\rightarrow \ker(A) = \{(t, -t, 0) \mid t \in \mathbb{R}^n\}$

Since ker(A) is an infinitely long set, multiple elements map to 0, breaking injectivily.

A is surjective because

im(A) = span {nonredundant columns of A}

(using rref to find corresponding nonredundant columns)

$$\mathrm{im}(A) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}\right\}$$

Since these vectors are all linearly independent (and thus the matrix formed by placing them side-by-side is invertible), they form a basis for $\mathbb{R}^3 \rightarrow \mathrm{im}(A) = \mathbb{R}^3 \rightarrow$ A is surjective.

Since A is NOT injective, it is NOT invertible.

c. $x_1 = x_4 - 2,\ x_2 = 2 - x_4,\ x_3 = 1$

$\rightarrow \{(t-2, 2-t, 1) \mid t \in \mathbb{R}\}$

## 1 Question 1 **8 / 10**

+ **10 pts** Correct
+ **4 pts** (a) correct
+ ✓ + **3 pts** **(a) one mistake**
+ **2 pts** (a) two or more mistakes, or a more serious mistake.
+ ✓ + **1 pts** **(b) not injective**
+ ✓ + **1 pts** **(b) surjective**
+ ✓ + **1 pts** **(c) not invertible**
- **1 pts** (b) did not explain fully

it more difficult to get this deduction if full points were not otherwise awarded for (b)

+ **3 pts** (c) correct
+ ✓ + **2 pts** **(c) one mistake**
+ **1 pts** (c) two mistakes or more serious mistakes.

1 Missing a column

2 No, the kernel is a subset of R^4; should be (t, -t, 0, t)

2. Let $A$ be the $3 \times 3$ matrix
$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & 3 & 1 \end{pmatrix}.$$

(a) (4 points) Find a basis for the kernel of the corresponding linear transformation.

(b) (3 points) Find a basis for the image of the corresponding linear transformation.

(c) (3 points) Geometrically describe the kernel and the image.

a. $\vec{x} \in \ker(A) \iff A\vec{x} = 0$

rref(A):

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{rref}(A)$$

$x_1 = \frac{-1}{3} x_3, \quad x_2 = \frac{-1}{3} x_3$

$\rightarrow \{(-\frac{1}{3}t, \frac{-1}{3}t, t) \mid t \in \mathbb{R}\} = \{t(-\frac{1}{3}, -\frac{1}{3}, 1) \mid t \in \mathbb{R}\}$

basis of $\ker(A) = \left\{ \begin{bmatrix} -1/3 \\ -1/3 \\ 1 \end{bmatrix} \right\}$ ← $t=0$

b. In rref(A), cols 1 & 2 are the only nonredundant columns ⇒ thus, in A, cols 1 & 2 are also the only nonredundant columns.

$\rightarrow \mathrm{im}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \right\}$

basis for $\mathrm{im}(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \right\}$

c. kernel: the LINE spanned by $\begin{bmatrix} -1/3 \\ -1/3 \\ 1 \end{bmatrix}$. (dimension 1)

image: the PLANE spanned by $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$ (dim. 2)

## 2 Question 2 10 / 10

- ✓ **+ 1 pts** (a) Set up equation $A\vec{x} = \vec{0}$ or set up row reduction.
- ✓ **+ 1 pts** (a) Row reduced $A$ correctly, or solved system correctly
- ✓ **+ 1 pts** (a) Given answer spans the kernel
- ✓ **+ 1 pts** (a) Given answer is linearly indepedent in $\mathbb{R}^3$
- ✓ **+ 1 pts** (b) Said image is span of columns (or other correct formulation)
- ✓ **+ 1 pts** (b) Given answer spans image
- ✓ **+ 1 pts** (b) Given answer is linearly independent in $\mathbb{R}^3$
- ✓ **+ 1 pts** (c) Kernel is a line in $\mathbb{R}^3$
- ✓ **+ 1 pts** (c) Image is a plane in $\mathbb{R}^3$
- ✓ **+ 1 pts** (c) Kernel and image pass through the origin, or an explicit description of which line and plane correspond to the kernel and image.
- **+ 0 pts** Blank or incorrect

3. (a) (5 points) Let $\vec{e_1}, \vec{e_2}, \vec{e_3}$ be the standard basis vectors in $\mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation that maps $e_1$ to $e_2$, maps $e_2$ to $e_1$, and maps the vector $(1, 0, 1)$ to $e_3$. Write down a matrix $A$ such that $T$ acts by multiplication by $A$ on the left, i.e. $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^3$.

   (b) (5 points) (This part is unrelated to part (a).) Write down a basis for the solution set to the linear system consisting of just the single equation

$$x_1 + 2x_2 + 3x_3 = 0.$$

a. $T(\vec{e}_1) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\vec{e}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \vec{e}_1$$

$$T(\vec{e}_3) = T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) - T(\vec{e}_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} | & | & | \\ T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \\ | & | & | \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$$

b. $x_1 + 2x_2 + 3x_3 = 0$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\hookrightarrow$ $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \vec{x} = 0$

Let $A = [1 \ 2 \ 3]$. $A\vec{x} = 0 \Leftrightarrow \vec{x} \in \ker(A)$

* That is: solution set $= \ker(A)$

$\mathrm{rref}(A) = [1 \ 2 \ 3]$

$\to x_1 = -2x_2 - 3x_3$

$\to \ker(A) = \{(-2t - 3s, t, s) \mid t, s \in \mathbb{R}\}$

$= \{t \underline{(-2, 1, 0)} + \underline{s(-3, 0, 1)} \mid t, s \in \mathbb{R}\}$

$\to$ basis for solution set $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

## 3 Question 3 **10 / 10**

✓ **+ 2 pts** **a) full credit: Starting approach of using linearity to deduce T(e_3), or finding T^{-1} to deduce T**

✓ **+ 3 pts** **a) full credit: Putting T(e_j) into columns of matrix, or using Gaussian elimination to find T from T^{-1}**

**+ 1 pts** a) partial credit: Some kind of approach given that leads to the right answer, but not clearly explained; or, linearity appears to be used, but it is not completely clear

**+ 1 pts** a) partial credit: formula for A in terms of T(e_j) correct, but incorrectly executed

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**+ 1 pts** a) partial credit: Some kind of attempt that vaguely makes sense, but is still incorrect and doesn't lead anywhere near the correct answer

**- 1 pts** computation error

**- 1 pts** computation error

✓ **+ 3 pts** **b) full credit: correct answer**

✓ **+ 2 pts** **b) full credit: justifying work**

**+ 1 pts** b) partial credit: some but not sufficient justification

**+ 1 pts** b) partial credit: partially correct work that leads to the wrong answer

4. Let $A$ be an $n \times m$ matrix, and let $\vec{b}$ be a vector in $\mathbb{R}^n$. Let $V_{\vec{b}}$ denote the solution set to $A\vec{x} = \vec{b}$. That is, we let

$$V_{\vec{b}} = \{\vec{x} \in \mathbb{R}^m \mid A\vec{x} = \vec{b}\}.$$

(We read the above as "the set of all $\vec{x}$ contained in $\mathbb{R}^m$ such that $A\vec{x} = \vec{b}$.")

(a) (3 points) Prove that $V_{\vec{b}}$ fails all three criteria for being a subspace if $\vec{b} \neq \vec{0}$.

(b) (3 points) Prove that if $V_{\vec{b}}$ is not the empty set, i.e. if $V_{\vec{b}}$ contains at least one vector denoted $\vec{x}$, then

$$V_{\vec{b}} = \vec{x} + \ker(A),$$

where $\vec{x} + \ker(A) = \{\vec{x} + \vec{y} \mid y \in \ker(A)\}$. In other words, $V_{\vec{b}}$ is equal to $\ker(A)$ shifted by the vector $\vec{x}$.

(c) (2 points) If $\ker(A)$ is a line, what is the size of a basis of $\text{im}(A)$? That is, what is the number of vectors in a basis for $\text{im}(A)$? (Hint: your answer could possibly depend on $n$ and/or $m$).

(d) (2 points) If $\ker(A)$ is a line, is $A$ injective? Surjective? Invertible? For each of these answer either "always", "sometimes" or "never."

a. 1. Criteria 1: $\vec{0} \in V_{\vec{b}}$

$A\vec{0} = \vec{0}$ (no matter what $A$ is)

if $\vec{b} \neq 0$, $\vec{A}\vec{0} \neq \vec{b}$, and $\vec{0} \notin V_{\vec{b}}$. ↯

2. Criteria 2: $\forall \vec{v}_1, \vec{v}_2 \in V_{\vec{b}}$, $\vec{v}_1 + \vec{v}_2 \in V_{\vec{b}}$

Let $\vec{v}_1, \vec{v}_2 \in V_{\vec{b}}$, then

$A\vec{v}_1 = \vec{b}$ and $A\vec{v}_2 = \vec{b}$

$\rightarrow$ $A\vec{v}_1 + A\vec{v}_2 = 2\vec{b}$

$A(\vec{v}_1 + \vec{v}_2) = 2\vec{b} \neq \vec{b}$ (by linearity of M-V multiplication).

$\therefore$ $\vec{v}_1 + \vec{v}_2 \notin V_{\vec{b}}$. ↯

3. Criteria 3: $\forall \vec{v} \in V_{\vec{b}}$, $\forall k \in \mathbb{R}$, $k\vec{v} \in V_{\vec{b}}$

Let $\vec{v} \in V_{\vec{b}}$ and $k \in \mathbb{R}$, then

$A\vec{v} = \vec{b}$

$\rightarrow$ $k(A\vec{v}) = k\vec{b}$ $\rightarrow$ $A(k\vec{v}) = k\vec{b} \neq \vec{b}$

$\therefore$ $(k\vec{v}) \notin V_{\vec{b}}$. ↯

$\boxtimes$

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b. Let $\vec{y} \in \ker(A)$, and let $\vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{b}$. Then,

$$A\vec{y} = 0$$

$$\rightarrow A\vec{y} + A\vec{x} = A\vec{x} = \vec{b}$$

$$A(\vec{y} + \vec{x}) = \vec{b} \qquad \text{(again by linearity)}$$

Thus, $\vec{x} + \vec{y} \in V_{\vec{b}}$. Next, we have to show $V_{\vec{b}}$ is comprised EXCLUSIVELY of elements in this form. Let $\vec{v} \in V_{\vec{b}}$ and the same $\vec{x}$ as above. $(\vec{z} = \vec{v} - \vec{x})$

$$A\vec{v} = \vec{b} \rightarrow A(\vec{x} + \vec{z}) = \vec{b} \rightarrow A\vec{x} + A\vec{z} = A\vec{x}$$

$$\rightarrow A\vec{z} = 0 \rightarrow \vec{z} \in \ker(A)$$

That is, every element in $V_{\vec{b}}$ is of the form $\vec{x} + \vec{y}$ for $\vec{y} \in \ker(\vec{A})$ and $A\vec{x} = \vec{b}$, and every possible element of this form is an element of $V_{\vec{b}}$, so

$$V_{\vec{b}} = \{\vec{x} + \vec{y} \mid \vec{y} \in \ker(\vec{A})\} \text{ where } A\vec{x} = \vec{b}. \qquad \boxtimes$$

c. $\dim(\ker(A)) = 1 = m - \dim(\operatorname{im}(A))$

$\dim(\operatorname{im}(A)) = m-1$

d. If $\ker(A)$ is a line, A is never injective because multiple distinct elements map to the same value $(\vec{0})$. A is SOMETIMES surjective (i.e. if $m = 0$, it is surjective! but if $m > n$ it is not). It is never invertible since it is never injective.

## 4 Question 4 10 / 10

✓ **+ 3 pts** (a):

(1): show $$V_{\vec{b}}$$ does not contain $$\vec{0}$$.
Proof: $$A\vec{0} = \vec{0}$$, but $$\vec{0} \neq \vec{b}$$ by assumption; so $$\vec{0}\notin V_{\vec{b}}$$ by definition

(2): show $$V_{\vec{b}}$$ is not closed under vector addition.
Proof: pick $$\vec{v} \in V_{\vec{b}}$$; then $$A(\vec{v}) = \vec{b}$$; but $$A(\vec{v}+\vec{v}) =\vec{b}+\vec{b}= 2\vec{b}$$, which is not equal to $$\vec{b}$$ since $$\vec{b}\neq \vec{0}$$ (by assumption); so $$\vec{v}+\vec{v} \notin V_{\vec{b}}$$

(3): show $$V_{\vec{b}}$$ is not closed under scalar multiplication.
Proof: pick $$\vec{v} \in V_{\vec{b}}$$; then $$A(\vec{v}) = \vec{b}$$; but $$A(-1\vec{v}) = - \vec{b}$$, which is not equal to $$\vec{b}$$ since $$\vec{b}\neq \vec{0}$$ (by assumption); so $$-1\vec{v} \notin V_{\vec{b}}$$

\[ We assume $$V_{\vec{b}}$$ is nonempty for (2) and (3); however, if the set $$V_{\vec{b}}$$ is empty, then it is not a subspace in this case either ]

✓ **+ 1 pts** (b): show $$(\vec{x}+\mathrm{ker}A) \subseteq V_{\vec{b}}$$, assuming $$\vec{x} \in V_{\vec{b}}$$.
Proof: let $$\vec{y} \in \mathrm{ker}A$$ be given. Then $$A\vec{y}=\vec{0}$$, and $$A\vec{x}=\vec{b}$$ already (by assumption);
so $$A(\vec{x}+\vec{y}) = A\vec{x}+A\vec{y} = \vec{0}+\vec{b}=\vec{b}$$,
so $$\vec{x}+\vec{y} \in V_{\vec{b}}$$.

✓ **+ 2 pts** (b): show $$V_{\vec{b}} \subseteq(\vec{x}+\mathrm{ker}A)$$, assuming $$\vec{x} \in V_{\vec{b}}$$.
Proof: let $$\vec{u} \in V_{\vec{b}} $$ be given. Then $$A\vec{u}=\vec{b}$$, and already $$A\vec{x} = \vec{b}$$ (by assumption).
Define $$\vec{y} = \vec{u}-\vec{x}$$; now $$A\vec{y} = A(\vec{u}-\vec{x})=A\vec{u}-A\vec{x}=\vec{b}-\vec{b}=\vec{0}$$; therefore $$\vec{y} \in \mathrm{ker}A$$.
So now $$\vec{u} = \vec{x}+(\vec{u}-\vec{x})= \vec{x}+\vec{y}$$, and $$\vec{y} \in \mathrm{ker}A$$; therefore $$\vec{u} \in \vec{x}+\mathrm{ker}A$$.

✓ **+ 2 pts** (c): basis for $$\mathrm{im}A$$ has $$m-1$$ vectors (by e.g. Rank Theorem; no justification needed in answer for this part)

✓ **+ 1 pts** (d): $$A$$ never injective (because kernel is nontrivial (since we are told the kernel is a line and therefore has dimension $$1>0$$));
so also never invertible (because invertible is equivalent to being injective and surjective simultaneously)
(no justification needed for this part)

✓ **+ 1 pts** (d): $$A$$ sometimes surjective. (no justification needed for this part)
In fact, in the case of this problem (if $$\mathrm{ker}A$$ is a line and $$A$$ is an $$n\times m$$ matrix), $$A$$ will be surjective iff $$n=m-1$$.