1. Consider the following system of linear equations.

$$x_1 + x_2 + x_3 = 1$$

$$x_1 + x_2 + 2x_3 = 2$$

$$x_2 + x_4 = 2$$

(a) (4 points) Rewrite the system as a matrix equation $A\vec{x} = \vec{b}$ and convert the augmented matrix $[A\,|\,\vec{b}]$ to reduced row echelon form.

(b) (3 points) Is the linear transformation T corresponding to A injective? Surjective? Invertible?

(c) (3 points) Write down the solution set to the linear system.

a) $$\boxed{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R2-R1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R1-R3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R1-R2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right] \rightarrow \boxed{\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]}$$

b) A is not a square matrix so it is not invertable

A is not injective because there is more than one solution $\vec{x}$ when $A\vec{x} = \vec{b}$. This is seen as rank(A) = 3 so there is 1 free variable.

A is surjective as T maps $\mathbb{R}^4 \to \mathbb{R}^3$ and the image is the span of $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right\}$, which is all of $\mathbb{R}^3$. This is also shown by # of pivots = # of rows.

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c) $x_1 - x_4 = -2$ $\quad$ x_4 is free

$x_2 + x_4 = 2$

$x_3 = 1$

$t = x_4$

$x_1 - t = -2$ $\quad$ $x_1 = t - 2$

$x_2 + t = 2$ $\quad$ $x_2 = 2 - t$

$x_3 = 1$

$$\boxed{\{(t-2,\ 2-t,\ 1,\ t) \mid t \in \mathbb{R}\}}$$

1 Question 1 **10 / 10**

- \+ **10 pts** Correct
- ✓ + **4 pts** **(a) correct**
- \+ **3 pts** (a) one mistake
- \+ **2 pts** (a) two or more mistakes, or a more serious mistake.
- ✓ + **1 pts** **(b) not injective**
- ✓ + **1 pts** **(b) surjective**
- ✓ + **1 pts** **(c) not invertible**
- \- **1 pts** (b) did not explain fully

it more difficult to get this deduction if full points were not otherwise awarded for (b)

- ✓ + **3 pts** **(c) correct**
- \+ **2 pts** (c) one mistake
- \+ **1 pts** (c) two mistakes or more serious mistakes.

2. Let A be the 3×3 matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & 3 & 1 \end{pmatrix}.$$

(a) (4 points) Find a basis for the kernel of the corresponding linear transformation.

(b) (3 points) Find a basis for the image of the corresponding linear transformation.

(c) (3 points) Geometrically describe the kernel and the image.

a) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{R2/2} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{R2-R1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix} \text{2S} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{R1 - \frac{2}{3}R2} \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R3/3} \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 + \frac{x_3}{3} = 0 \quad x_2 + \frac{x_3}{3} = 0$

$x_1 = -\frac{x_3}{3} \quad x_2 = -\frac{x_3}{3}$

$x_3 = 1 \quad \left(-\frac{1}{3}, -\frac{1}{3}, 1\right)$

$\boxed{\text{basis}(\ker(A)) = \begin{bmatrix} -1/3 \\ -1/3 \\ 1 \end{bmatrix}}$

b) colums 1 & 2 are linearly independent

$\text{im}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}\right)$

$\boxed{\text{basis}(\text{im}(A)) = \left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}\right\}}$

c) kernel: a line in $\mathbb{R}^3$ through the origin

image: A plane in $\mathbb{R}^3$ through the origin

2 Question 2 **10 / 10**

- ✓ + **1 pts** (a) Set up equation $$A\vec{x} = \vec{0}$$ or set up row reduction.
- ✓ + **1 pts** (a) Row reduced $$A$$ correctly, or solved system correctly
- ✓ + **1 pts** (a) Given answer spans the kernel
- ✓ + **1 pts** (a) Given answer is linearly indepedent in $$\mathbb{R}^3$$
- ✓ + **1 pts** (b) Said image is span of columns (or other correct formulation)
- ✓ + **1 pts** (b) Given answer spans image
- ✓ + **1 pts** (b) Given answer is linearly independent in $$\mathbb{R}^3$$
- ✓ + **1 pts** (c) Kernel is a line in $$\mathbb{R}^3$$
- ✓ + **1 pts** (c) Image is a plane in $$\mathbb{R}^3$$
- ✓ + **1 pts** (c) Kernel and image pass through the origin, or an explicit description of which line and plane correspond to the kernel and image.
- + **0 pts** Blank or incorrect

3. (a) (5 points) Let $\vec{e_1}, \vec{e_2}, \vec{e_3}$ be the standard basis vectors in $\mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation that maps e_1 to e_2, maps e_2 to e_1, and maps the vector $(1, 0, 1)$ to e_3. Write down a matrix A such that T acts by multiplication by A on the left, i.e. $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^3$.

(b) (5 points) (This part is unrelated to part (a).) Write down a basis for the solution set to the linear system consisting of just the single equation

$$x_1 + 2x_2 + 3x_3 = 0.$$

a) $T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$

$T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$T\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$$\begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$T\begin{bmatrix}0\\0\\1\end{bmatrix} = T\begin{bmatrix}1\\0\\1\end{bmatrix} - T\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix}0\\0\\1\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$$\boxed{A = \begin{bmatrix}0 & 1 & 0\\1 & 0 & -1\\0 & 0 & 1\end{bmatrix}}$$

b) $x_1 = -3x_3 - 2x_2$

x_3	x_2	
1	0	$(-3, 0, 1)$
0	1	$(-2, 1, 0)$

$$\boxed{\text{basis} = \left\{\begin{bmatrix}-3\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\0\end{bmatrix}\right\}}$$

3 Question 3 **10 / 10**

✓ **+ 2 pts** **a) full credit: Starting approach of using linearity to deduce T(e_3), or finding T^{-1} to deduce T**

✓ **+ 3 pts** **a) full credit: Putting T(e_j) into columns of matrix, or using Gaussian elimination to find T from T^{-1}**

+ 1 pts a) partial credit: Some kind of approach given that leads to the right answer, but not clearly explained; or, linearity appears to be used, but it is not completely clear

+ 1 pts a) partial credit: formula for A in terms of T(e_j) correct, but incorrectly executed

+ 1 pts a) partial credit, first two columns of matrix correct, but no clearly written correct reasoning

+ 1 pts a) partial credit: Some kind of attempt that vaguely makes sense, but is still incorrect and doesn't lead anywhere near the correct answer

- 1 pts computation error

- 1 pts computation error

✓ **+ 3 pts** **b) full credit: correct answer**

✓ **+ 2 pts** **b) full credit: justifying work**

+ 1 pts b) partial credit: some but not sufficient justification

+ 1 pts b) partial credit: partially correct work that leads to the wrong answer

4. Let A be an $n \times m$ matrix, and let $\vec{b}$ be a vector in $\mathbb{R}^n$. Let $V_{\vec{b}}$ denote the solution set to $A\vec{x} = \vec{b}$. That is, we let

$$V_{\vec{b}} = \{\vec{x} \in \mathbb{R}^m \mid A\vec{x} = \vec{b}\}.$$

(We read the above as "the set of all $\vec{x}$ contained in $\mathbb{R}^m$ such that $A\vec{x} = \vec{b}$.")

(a) (3 points) Prove that $V_{\vec{b}}$ fails all three criteria for being a subspace if $\vec{b} \neq \vec{0}$.

(b) (3 points) Prove that if $V_{\vec{b}}$ is not the empty set, i.e. if $V_{\vec{b}}$ contains at least one vector denoted $\vec{x}$, then

$$V_{\vec{b}} = \vec{x} + \ker(A),$$

where $\vec{x} + \ker(A) = \{\vec{x} + \vec{y} \mid y \in \ker(A)\}$. In other words, $V_{\vec{b}}$ is equal to $\ker(A)$ shifted by the vector $\vec{x}$.

(c) (2 points) If $\ker(A)$ is a line, what is the size of a basis of $\mathrm{im}(A)$? That is, what is the number of vectors in a basis for $\mathrm{im}(A)$? (Hint: your answer could possibly depend on n and/or m).

(d) (2 points) If $\ker(A)$ is a line, is A injective? Surjective? Invertible? For each of these answer either "always", "sometimes" or "never."

a) 1. If $\vec{b} \neq \vec{0}$, then $\vec{0}$ is not in $V_{\vec{b}}$, so it is not a subspace

2. It is not closed under addition. If $A\vec{w_1} = \vec{b}$ $A\vec{w_2} = \vec{b}$ then $A(\vec{w_1} + \vec{w_2}) = 2\vec{b}$, which means that $\vec{w_1} + \vec{w_2}$ is not in $V_{\vec{b}}$.

3. For $A\vec{w} = \vec{b}$, $A(k\vec{w}) = k\vec{b}$, which shows that $k\vec{w}$ is not in $V_{\vec{b}}$

b) If $A\vec{x} = \vec{b}$ and $A\ker(A) = \vec{0}$ then $A(\vec{x} + \ker(A))$ $= A\vec{x} + A\ker(A) = A\vec{x} + \vec{0} = \vec{b} \Rightarrow A\vec{x} = \vec{b}$. This shows that for all $\vec{x}$, $A(\vec{x} + \ker(A)) = \vec{b}$, meaning that $\vec{x} + \ker(A)$ is the solution set to $A\vec{x} = \vec{b}$, which shows $V_{\vec{b}} = \vec{x} + \ker(A)$

c) $\dim(\ker(A)) = 1$ for a line. By rank nullity, $\dim(\ker(A)) = m - \mathrm{rank}(A)$ and $\dim(\mathrm{im}(A)) = \mathrm{rank}(A)$, so $\dim(\mathrm{im}(A)) = m - \dim(\ker(A)) = m-1$

so $\boxed{\text{size of basis of } \mathrm{im}(A) = m-1}$

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d) A is never injective as A can only be injective when $\dim(\ker(A)) = 0$, which is not true if the $\ker(A)$ is a line.

A is sometimes surjective as $\text{rank}(A) = m-1$ for a line. It will be surjective when $\text{rank}(A) = n$, or in other words, when $n = m-1$.

A is never invertible as A must be injective, which it never is as $\dim(\ker(A)) \neq 0$

4 Question 4 **8 / 10**

✓ **+ 3 pts** (a):

(1): show $$V_{\vec{b}}$$ does not contain $$\vec{0}$$.
Proof: $$A\vec{0} = \vec{0}$$, but $$\vec{0} \neq \vec{b}$$ by assumption; so $$\vec{0}\notin V_{\vec{b}}$$ by definition

(2): show $$V_{\vec{b}}$$ is not closed under vector addition.
Proof: pick $$\vec{v} \in V_{\vec{b}}$$; then $$A(\vec{v}) = \vec{b}$$; but $$A(\vec{v}+\vec{v}) =\vec{b}+\vec{b}= 2\vec{b}$$, which is not equal to $$\vec{b}$$ since $$\vec{b}\neq \vec{0}$$ (by assumption); so $$\vec{v}+\vec{v} \notin V_{\vec{b}}$$

(3): show $$V_{\vec{b}}$$ is not closed under scalar multiplication.
Proof: pick $$\vec{v} \in V_{\vec{b}}$$; then $$A(\vec{v}) = \vec{b}$$; but $$A(-1\vec{v}) = - \vec{b}$$, which is not equal to $$\vec{b}$$ since $$\vec{b}\neq \vec{0}$$ (by assumption); so $$-1\vec{v} \notin V_{\vec{b}}$$

\[We assume $$V_{\vec{b}}$$ is nonempty for (2) and (3); however, if the set $$V_{\vec{b}}$$ is empty, then it is not a subspace in this case either]

✓ **+ 1 pts** (b): show $$(\vec{x}+\mathrm{ker}A) \subseteq V_{\vec{b}}$$, assuming $$\vec{x} \in V_{\vec{b}}$$.
Proof: let $$\vec{y} \in \mathrm{ker}A$$ be given. Then $$A\vec{y}=\vec{0}$$, and $$A\vec{x}=\vec{b}$$ already (by assumption);
so $$A(\vec{x}+\vec{y}) = A\vec{x}+A\vec{y} = \vec{0}+\vec{b}=\vec{b}$$,
so $$\vec{x}+\vec{y} \in V_{\vec{b}}$$.

+ 2 pts (b): show $$V_{\vec{b}} \subseteq(\vec{x}+\mathrm{ker}A)$$, assuming $$\vec{x} \in V_{\vec{b}}$$.
Proof: let $$\vec{u} \in V_{\vec{b}} $$ be given. Then $$A\vec{u}=\vec{b}$$, and already $$A\vec{x} = \vec{b}$$ (by assumption).
Define $$\vec{y} = \vec{u}-\vec{x}$$; now $$A\vec{y} = A(\vec{u}-\vec{x})=A\vec{u}-A\vec{x}=\vec{b}-\vec{b}=\vec{0}$$; therefore $$\vec{y} \in \mathrm{ker}A$$.
So now $$\vec{u} = \vec{x}+(\vec{u}-\vec{x})= \vec{x}+\vec{y}$$, and $$\vec{y} \in \mathrm{ker}A$$; therefore $$\vec{u} \in \vec{x}+\mathrm{ker}A$$.

✓ **+ 2 pts** (c): basis for $$\mathrm{im}A$$ has $$m-1$$ vectors (by e.g. Rank Theorem; no justification needed in answer for this part)

✓ **+ 1 pts** (d): $$A$$ never injective (because kernel is nontrivial (since we are told the kernel is a line and therefore has dimension $$1>0$$));
so also never invertible (because invertible is equivalent to being injective and surjective simultaneously)
(no justification needed for this part)

✓ **+ 1 pts** (d): $$A$$ sometimes surjective. (no justification needed for this part)
In fact, in the case of this problem (if $$\mathrm{ker}A$$ is a line and $$A$$ is an $$n\times m$$ matrix), $$A$$ will be surjective iff $$n=m-1$$.

1 technically here the assumption $$k\neq 1$$ is also needed