Math 33A **Name:** ____________________

ID Number ____________________

Spring 2020
Final Exam

This exam contains 17 pages (including this cover page) and 8 problems.

Important Notice: Academic dishonesty is a very serious offense, and the Office of the Dean of Students is charged with investigating and adjucating suspected violations. The penalties for academic dishonesty are often very severe.

You have 48 hours to complete this exam. You are required to show your work on each problem on this exam, unless otherwise stated. This holds even if work isn't explicitly asked for in the statement of the problem. Please take care to show your steps, providing a reasonable amount of justification for your work that demonstrates your understanding.

The following rules apply:

- You don't need to print this out and write directly on the exam, you may submit separate sheets of paper. If you print out and write on the exam, you may use the provided blank pages for your work. In either case, make sure you organize your work, in a reasonably neat and coherent way. Work scattered all over the page without a clear ordering will receive very little credit.
- In problems that have multiple parts, if your answer for a later part depends on a previous part, you can still get partial credit for the later part even if your answer for the previous part is incorrect.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

1. (a) (5 points) Find the set of solutions of the linear system

$$x_1 + 2x_2 + 3x_3 - 4x_4 = 1$$

$$x_1 + 2x_2 + 3x_3 + 2x_4 = 2$$

$$2x_1 + 4x_2 + 6x_3 + 4x_4 = 4$$

(b) (3 points) Write the above linear system in matrix form $A\vec{x} = \vec{b}$ and compute rank(A). Describe the solution set geometrically.

(c) (2 points) Suppose that the solution set to one given linear system of equations is a plane, and the solution set to another given linear system of equations is also a plane. What are the possible solution sets of the system of linear equations obtained by combining the equations in both systems?

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2. (a) (1 point) Give an example of a linear transformation that is injective but not surjective.

   (b) (1 point) Give an example of a linear transformation that is surjective but not injective.

   (c) (3 points) Suppose that T is an injective linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ and that S is an injective linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^p$. Prove that composition $R = S \circ T$ defined by $R(\vec{x}) = S(T(\vec{x}))$ is an injective linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^p$.

   (d) (5 points) Find the matrix for the linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$ that maps $(1, 2, 3, 4)$ to $(2, 2, 3, 4)$, maps e_2 to e_3, maps e_3 to e_2, and maps $(1, 0, 0, 1)$ to e_4.

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3. Let V be a subspace of $\mathbb{R}^n$, and let P be the orthogonal projection matrix onto V. That is, P satisfies

$$P\vec{x} = \text{proj}_V \vec{x}, \qquad \text{for all } \vec{x} \text{ in } \mathbb{R}^n.$$

   (a) (2 points) What are the eigenvalues of P? Justify your answer.

   (b) (3 points) What are the corresponding eigenspaces? Justify your answer.

   (c) (2 points) Can P be diagonalized? Justify your answer.

   (d) (3 points) Prove that $P^2 = P$.

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4. (a) (4 points) Let A be the 2×2 matrix of rotation by 45 degrees counterclockwise in the plane. Let $[A]_{\mathcal{B}}$ denote the $\mathcal{B}$-matrix of A with respect to the basis $\{(1,1),(0,1)\}$. Find $[A]_{\mathcal{B}}$.

   (b) (3 points) Let $\mathcal{B}$ be a basis of $\mathbb{R}^n$. Prove that

$$[AB]_{\mathcal{B}} = [A]_{\mathcal{B}}[B]_{\mathcal{B}}.$$

   That is, prove that the $\mathcal{B}$-matrix of AB is equal to the $\mathcal{B}$-matrix of A times the $\mathcal{B}$-matrix of B.

   (c) (3 points) Suppose that A is a rotation matrix in $\mathbb{R}^3$, and suppose that $\mathcal{B}$ is some basis of $\mathbb{R}^3$. What is the determinant of the $\mathcal{B}$-matrix of A? Justify your answer.

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5. (a) (5 points) Suppose that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix A maps the parallelogram formed by the vectors

$$\{(1,5),(1,0)\}$$

to the parallelogram formed by the vectors

$$\{(2,3),(4,5)\}.$$

If Q is a square of sidelength $\sqrt{2}$, what is the area of the image of Q under the linear transformation $(A^T A)^{-1}$? (That is, what is the area of $(A^T A)^{-1}(Q)$?)

(b) (2 points) Find the determinant of the 4×4 matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 4 & 1 \end{pmatrix}$$

Be sure to show your work. An answer without the relevant work shown will receive very little credit.

(c) (3 points) Find the determinant of the 7×7 matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{pmatrix}$$

Be sure to show your work. An answer without the relevant work shown will receive very little credit.

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6. Define a quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$ by

$$q(x_1, x_2, x_3) = 3x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_3$$

(a) (2 points) Find the associated 3×3 symmetric matrix A such that $q(\vec{x}) = \vec{x} \cdot (A\vec{x})$.

(b) (3 points) Find an orthogonal diagonalization of A.

(c) (1 point) What is the definiteness of q?

(d) (4 points) Use the information from part(b) to geometrically describe the level set

$$\{\vec{x} \in \mathbb{R}^3 : q(\vec{x}) = 10\},$$

read as "the set of all vectors $\vec{x}$ in $\mathbb{R}^3$ such that $q(\vec{x}) = 10$." Be sure to explain how you use the information from part(b). You can sketch the level set if you'd like but it is not required.

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7. (a) (5 points) Find a 5x5 matrix A whose corresponding linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$ satisfies **all** of the following four criteria, $(i), (ii), (iii), (iv)$:

   (i) $T((1, 2, 3, 4, 5)) = (1, 2, 3, 4, 5)$,
   (ii) $T(\vec{e_1}) = -\vec{e_1}$,
   (iii) $T(\vec{e_2}) = 10\vec{e_2}$,
   (iv) $\dim(\ker(T)) = \dim(\ker(A)) = 2$, i.e., the dimension of the kernel of A is 2.

   (b) (5 points) Compute A^{2020}, being sure to show your work.

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8. Let

$$B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$$

(a) (5 points) Find the singular values of the matrix B given above. Show all your steps. An answer given without the relevant work will receive very little credit.

(b) (5 points) Find a singular value decomposition $B = U\Sigma V^T$ of the matrix B. Show all your steps. An answer given without the relevant work will receive very little credit.

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