20S-MATH33A-2 Final Exam

FRANK ZHENG

TOTAL POINTS

78 / 80

QUESTION 1

1 Question 1 10 / 10

✓ + **5 pts** **a) Full credit: Convert the augmented system into rref form and get \[(1, 2, 3, 0); (0, 0, 0, 1); (0, 0, 0, 0)| 5/3; 1/6, 0]. Two free variables, x_2:=t, x_3:=s. Solution set is then {(5/3-2t-3s, t, s, 1/6)| s, t, real numbers}.**

✓ + **3 pts** **b) Full credit: The rank is 2 as there are 2 pivots. The solution set is a plane in R^4.**

✓ + **2 pts** **c) Full credit: The solution set of the combined system is the intersection of the two planes; it could possibly be empty (if the planes don't intersect) a point (for example, if the planes are in R^4 for instance and orthogonal complements of each other), a line, or a plane (if the two planes are the same).**

+ **3 pts** a) partial credit: convert to rref

+ **1 pts** a) partial credit: label free variables

+ **1 pts** a) partial credit: express solution set in terms of free variables (or alternatively find a basis)

+ **1.5 pts** b) partial credit: only one of rank=2 and plane given correctly

+ **1 pts** c) Partial credit: part of the solution, or some things correct and some incorrect

+ **0 pts** c) Mostly incorrect

QUESTION 2

2 Question 2 10 / 10

✓ + **1 pts** **a) Injective but not surjective**

✓ + **1 pts** **b) surjective but not injective**

✓ + **3 pts** **c) Full credit: If x is a nonzero element of Rm then T injective implies T(x) is nonzero, and then S injective implies S(T(x)) is nonzero, and so R(x) is nonzero. Therefore R is injective, as its kernel contains only the zero vector. (Not needed as this was already shown in class, but we know R is a linear transformation since R(x+y)=S(T(x)+T(y))=S(T(x))+S(T(y))=R(x)+R(y) by the linearity of S and T, and also for any scalar c, R(c(x))=S(c(T(x))=c(S(T(x))=x(R(x)) again by the linearity of S and T.)**

✓ + **5 pts** **d) Full credit: Correct answer given by (-2/3, 0, 0, 2/3; 1/3, 0, 1, -1/3; -1/3, 1, 0, 1/3; 0, 0, 0, 1). Show work by either using change of basis matrix, or by computing 3T(e_4)=T(1, 2, 3, 4)-T(1, 0, 0, 1)-2T(e_2)-3T(e_3)=(2, 2, 3, 4)-e_4-2e_3-3e_2=(2, -1, 1, 3) which implies T(e_4)=(2/3, -1/3, 1/3, 1) and by computing T(e_1)=T(1, 0, 0, 1)-T(e_4)=e_4-(2/3, -1/3, 1/3, 1) =(-2/3, 1/3, -1/3, 0)**

+ **2 pts** c) Partial credit: Nearly correct proof that is missing a minor justification

+ **1 pts** c) Partial credit: A reasonable attempt that does not give a valid proof, but still makes use of the definition of injectivity of S and T to conclude something about the injectivity of R.

+ **3 pts** d) Partial credit: Correct answer with some but insufficient justification

+ **0 pts** 0 points

QUESTION 3

3 Question 3 10 / 10

✓ + **1 pts** **(a): show 0,+1 are possible eigenvalues of P.**

Solution:

For nonzero $$\vec{x} \in V$$ (assuming $$V \neq \{\vec{0}\}$$) we have $$P\vec{x} = \vec{x}$$ so $$x$$ has eigenvalue 1 for $$P$$; so 1 is an eigenvalue of $$P$$.

For nonzero $$\vec{x} \in V^{\perp}$$ (assuming $$V^{\perp} \neq \{\vec{0}\}$$) we have $$P\vec{x} = \vec{0}$$ so $$\vec{x}$$ has eigenvalue 0 for

$$P$$; so $$0$$ is an eigenvalue for $$P$$.

✓ + 1 pts (a): show no eigenvalues other than 0,+1 are possible for P.

Solution:

Let $$\lambda$$ be an eigenvalue of $$P$$; let $$\vec{x}\neq\vec{0}$$ be an eigenvector of $$P$$ with eigenvalue $$\lambda$$; so $$\lambda\vec{x} = P\vec{x}$$. Next, due to the decomposition $$\mathbb{R}^n = V \oplus V^{\perp}$$ we can (uniquely) pick $$\vec{v} \in V$$ and $$\vec{v}_{\perp} \in V^{\perp}$$ such that $$\vec{x} = \vec{v} + \vec{v}_{\perp}$$; then now $$P(\vec{x}) = P(\vec{v} + \vec{v}_{\perp}) = \vec{v}$$.

So now $$\vec{v} = P\vec{x} = \lambda \vec{x} = \lambda \vec{v} + \lambda \vec{v}_{\perp}$$, so $$(1-\lambda)\vec{v} = \lambda \vec{v}_{\perp}$$. But now the left-hand-side $$(1-\lambda)\vec{v}$$ lies in $$V$$ while the right-hand-side $$\lambda \vec{v}_{\perp}$$ lies in $$V^{\perp}$$; so both sides are $$\vec{0}$$ (since $$V \cap V^{\perp} = \{\vec{0}\}$$).

So now both $$(1-\lambda)\vec{v} = \vec{0}$$ and $$\lambda \vec{v}_{\perp} = \vec{0}$$.

Case 1: if $$\vec{v}_{\perp} \neq \vec{0}$$, we must have $$\lambda = 0$$. Case 2: if $$\vec{v}_{\perp} = \vec{0}$$ (which then implies $$\vec{v}\neq\vec{0}$$ since $$\vec{x}\neq\vec{0}$$ and $$\vec{x} = \vec{v}+\vec{v}_{\perp}$$), then we must have $$1-\lambda=0$$.

So $$\lambda \in \{0,+1\}$$ are the only possibilities.

\[Alternate method 1: use geometric argument. If $$\vec{x}\neq\vec{0}$$ is an eigenvector of $$P$$ with eigenvalue not zero, then $$P\vec{x}$$ is both nonzero and parallel to $$\vec{x}$$. Since $$P$$ is an orthogonal projection, it is only possible for $$\vec{x},P\vec{x}$$ to be nonzero and parallel to each other if $$\vec{x} \in V$$.]

\[Alternate method 2: use the fact that $$P^2 = P$$ to argue eigenvalues must satisfy $$\lambda^2 = \lambda$$, whose only solutions are $$\lambda = 0,+1$$]

\[Alternate method 3: show that the 0-eigenspace is $$V$$ and the +1-eigenspace is $$V^{\perp}$$, so the sum of these eigenspaces has dimension $$n$$, and therefore is all of $$\mathbb{R}^n$$; therefore no eigenvalues other than 0,+1 are possible.

This proof uses repeatedly the fact that any two eigenspaces (for distinct eigenvalues) (of the same matrix) intersect trivially (meaning their intersection is $$\{\vec{0}\}$$), and so the dimension of their sum is the sum of their dimensions]

✓ + 2 pts (b): conclude 0-eigenspace is $$V^{\perp}$$ and +1-eigenspace is $$V$$.

Proof:

Let $$E_0$$ and $$E_1$$ denote the eigenspaces of $$P$$ for eigenvalues $$0$$ and $$+1$$ respectively.

First, already $$P$$ acts as multiplication by 0 on $$V^{\perp}$$ and as multiplication by 1 on $$V$$. (Since if $$\vec{v} \in V$$ then $$P\vec{v} = \vec{v}$$, and if $$\vec{v}^{\perp} \in V^{\perp}$$ then $$P\vec{v}^{\perp} = \vec{0}$$.) This proves $$V^{\perp} \subseteq E_0$$ and $$V \subseteq E_1$$.

It only remains to show the converse: that eigenvectors of $$P$$ with eigenvalue 0 are in $$V^{\perp}$$, and that eigenvectors of $$P$$ with eigenvalue 1 are in $$V$$. (In other words, that $$E_0 \subseteq V^{\perp}$$ and that $$E_1 \subseteq V$$.)

\[Method 1]

Let $$\vec{x} \in \mathbb{R}^n$$ be a given eigenvector of $$P$$ (with $$\vec{x}\neq\vec{0}$$) with eigenvalue $$\lambda$$; so $$\lambda \vec{x} = P\vec{x}$$. Using $$\mathbb{R}^n = V \oplus

$V^{\perp}$ write $\vec{x} = \vec{v} + \vec{v}^{\perp}$ for some $\vec{v} \in V$ and some $\vec{v}^{\perp}\in V^{\perp}$.
From part (a) we must have $\lambda = 0$ or $\lambda = 1$.
We want to show that if $\lambda = 0$ then $\vec{x} \in V^{\perp}$, and that if $\lambda = 1$ then $\vec{x} \in V$.
Note $P\vec{x} = P(\vec{v}+\vec{v}^{\perp}) = \vec{v}$.
Case 1: if $\lambda = 0$, then $\vec{0} = 0\vec{x} = P\vec{x} = \vec{v}$; so $\vec{v}=\vec{0}$, therefore $\vec{x} = \vec{v}^{\perp} \in V^{\perp}$.
Case 2: if $\lambda = 1$, then $\vec{x} = 1\vec{x} = P\vec{x} = \vec{v}$; so $\vec{x} = \vec{v} \in V$.

This completes the proof that $V=E_1$ and $V^{\perp} = E_0$.

\[Method 2]

We have shown already that $V^{\perp} \subseteq E_0$ and $V \subseteq E_1$. It follows that $V{\perp} + V \subseteq E_0 + E_1$. But $V^{\perp}+V = \mathbb{R}^n$; it must be the case that $E_0 + E_1 = \mathbb{R}^n$.

Next, we know $E_0 \cap E_1 = \{\vec{0}\}$ since E_0,E_1 are eigenspaces with distinct eigenvalues; it follows that $\mathrm{dim}(E_0+E_1) = \mathrm{dim}E_0+\mathrm{dim}E_1$. But $E_0 + E_1 = \mathbb{R}^n$. So now we have $n = \mathrm{dim}E_0+\mathrm{dim}E_1$.

Let $k := \mathrm{dim}V^{\perp}$; so $\mathrm{dim}V = n-k$ (because $V\oplus V^{\perp} = \mathbb{R}^n$). It follows that $\mathrm{dim}E_0 \geq k$ and $\mathrm{dim}E_1 \geq n-k$ (since $E_0 \supseteq V^{\perp}$ and $E_1 \supseteq V$).

Substituting $\mathrm{dim}E_1 = n-\mathrm{dim}E_0$ in the second relation above gives $n-\mathrm{dim}E_0\geq n-k$, which is equivalent to $\mathrm{dim}E_0 \leq k$.

So we have found $\mathrm{dim}E_0\geq k$ and $\mathrm{dim}E_0 \leq k$; it follows that $\mathrm{dim}E_0 = k$. And since $\mathrm{dim}E_1 +\mathrm{dim}E_0=n$ we then have $\mathrm{dim}E_1 = n-k$.

Finally, since $V^{\perp} \subseteq E_0$ and $\mathrm{dim}V^{\perp} = k = \mathrm{dim}E_0$ we must have $E_0 = V^{\perp}$;
similarly, since $V\subseteq E_1$ and $\mathrm{dim}V = n-k = \mathrm{dim}E_1$ we must have $E_1 = V$.
✓ + 1 pts (b): showing work/justification/proof
✓ + 1 pts (c): correctly say P can be diagonalized
✓ + 1 pts (c): provide sufficient justification for diagonalizability.
(e.g. the eigenspaces of P span all of $\mathbb{R}^n$, since $\mathbb{R}^n = V^{\perp} \oplus V = E_0^{(P)} \oplus E_{+1}^{(P)}$)
(equivalently: the geometric multiplicities of the eigenspaces add to n)
(more precisely: we can diagonalize $P = SDS^{-1}$ where the columns of S consist of a basis for V followed by a basis for $V^{\perp}$, and D is diagonal with first 1's for the V basis and 0's for the $V^{\perp}$ basis.)

\[alternate method: show P is symmetric, then apply spectral theorem]
✓ + 2 pts (d): argue that $P^2 = P$.

\[Method 1]

We recall P is diagonalizable (as shown in part (c)), and its eigenvalues are only $0,1$.

It follows that we can write $P = SDS^{-1}$ where

$$S,D$$ are square matrices (of the same size as $$P$$) with $$S$$ invertible, and $$D$$ diagonal with all diagonal entries of $$D$$ being either 0 or 1.
In particular it follows that $$D^2 = D$$ (because $$D$$ is diagonal, and each diagonal entry $$t$$ of $$D$$ is either 0 or 1 and therefore satisfies $$t^2 = t$$).

Now $$P^2 = (SDS^{-1})^2 = (SDS^{-1})(SDS^{-1}) = SD(S^{-1}S)DS^{-1} = SDIDS^{-1} = S(D^2)S^{-1} = SDS^{-1} = P;$$ note we used $$D^2 = D$$.

\[Method 2]

Recall that $$P\vec{a} \in V$$ for any vector $$\vec{a} \in \mathbb{R}^n$$. Furthermore, we also know $$P\vec{v} = \vec{v}$$ for any $$\vec{v} \in V$$.

It follows that for any vector $$\vec{x} \in \mathbb{R}^n$$ we have $$P(P\vec{x}) = P\vec{x}$$; so $$P^2\vec{x} = P\vec{x}$$.

Since $$P^2$$ and $$P$$ agree on all vectors $$\vec{x} \in \mathbb{R}^n$$ it follows that $$P^2 = P$$.

✓ + 1 pts (d): showing work/justification

QUESTION 4

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✓ + 1 pts (a): correct matrix $$A = \begin{bmatrix}
\cos45^{\circ} & -\sin45^{\circ}\\
\sin45^{\circ} & \cos45^{\circ}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}$$

✓ + 1 pts (a): correct change-of-basis formula: $$[A]_{\mathcal{B}} = S^{-1}AS$$ where the columns of $$S$$ are the ordered basis vectors in $$\mathcal{B}$$ (or other method)

✓ + 1 pts (a): correctly identify $$S = \begin{bmatrix}
1 & 0\\
1 & 1
\end{bmatrix}$$, and compute $$S^{-1} = \begin{bmatrix}
1 & 0\\
-1 & 1
\end{bmatrix}$$

+ 0 pts temp

✓ + 1 pts (a): compute $$[A]_{\mathcal{B}} = \begin{bmatrix}
0 & -\frac{1}{\sqrt{2}}\\
\sqrt{2} & \sqrt{2}
\end{bmatrix}$$

✓ + 1 pts (b): correct change-of-basis formulas for $$[A]_{\mathcal{B}}$$, $$[B]_{\mathcal{B}}$$, $$[AB]_{\mathcal{B}}$$

✓ + 2 pts (b): directly compute: $$[A]_{\mathcal{B}} [B]_{\mathcal{B}} = (S^{-1}AS)(S^{-1}BS) = S^{-1}A(SS^{-1})BS = S^{-1}AIBS = S^{-1}(AB)S = [AB]_{\mathcal{B}}.$$
Here $$S$$ is the matrix whose columns are the vectors in $$\mathcal{B}$$ (in the same order as in $$\mathcal{B}$$).

✓ + 1 pts (c): argue that a rotation matrix has determinant $$\pm1$$.
For example, a geometric argument can be used, by applying the geometric definition of determinant. Rotations preserve lengths (as well as angles and volumes) so a rotation maps a cube to another cube of the same volume.

\[Specifically, the determinant is +1 for orientation-preserving rotations (known as proper rotations), and -1 for orientation-reversing rotations (known as improper rotations).]

\[Note: just +1 instead of ±1 will also be accepted, as the textbook defines a rotation as having determinant +1.]

Alternatively, we can argue rotation matrices are orthogonal matrices (because they preserve length), and orthogonal matrices must have determinant $$\pm1$$.

✓ + 2 pts (c): argue that $$A$$ and $$[A]_{\mathcal{B}}$$ have the same determinant.

Proof: $$\mathrm{det}([A]_{\mathcal{B}}) = \mathrm{det}(S^{-1}AS) = \mathrm{det}(S^{-1})\mathrm{det}(A)\mathrm{det}(S) = \mathrm{det}(A) \,\mathrm{det}(S)^{-1}\mathrm{det}(S) = \mathrm{det}(A).$$ Note here we used the multiplicative properties of determinant (i.e., $$\mathrm{det}(CC') = \mathrm{det}(C)\mathrm{det}(C')$$ for any square matrices $$C,C'$$ of the same size).
\[More generally, any two similar square matrices have the same determinant.]

Alternatively, we can argue that the determinant of a linear operator (or square matrix) can be calculated in a basis-independent way (i.e. the geometric definition of determinant, which looks at how the operator changes the volume of oriented parallelepipeds, etc), and so the determinant should be the same in any basis.

QUESTION 5

5 Question 5 8 / 10

✓ + 2 pts (a) Made a connection between areas and determinants

+ 1 pts (a) Computed $$A =\left[\begin{array}{cc} 4 & -2/5 \\ 5 & -2/5\end{array}\right]$$ or $$A =\left[\begin{array}{cc} 2 & 2/5 \\ 3 & 2/5\end{array}\right]$$, or its determinant of $$\pm 2/5$$

✓ + 1 pts (a) Used $$\det A = \det A^T$$ or $$\det A^{-1} = 1/\det A$$, or computed $$(A^TA)^{-1}$$ directly

+ 1 pts (a) Correct final answer: $$\displaystyle\frac{25}{2}$$

✓ + 1 pts (b) Correct determinant: $$4$$

✓ + 1 pts (b) Showed work, or partial credit for work towards a determinant

✓ + 1 pts (c) Correct determinant: $$0$$

✓ + 2 pts (c) Showed work, or partial credit for work towards a determinant

1 This should be 2/5, not 5/2

QUESTION 6

6 Question 6 10 / 10

✓ + 2 pts (a) Correct matrix: $$A = \left[\begin{array}{ccc} 3 & 0 & 2\\ 0 & 2 & 0\\ 2 & 0 & 3\end{array} \right]$$

✓ + 1 pts (b) Correct eigenvalues: $$\lambda = 1,2,5$$

✓ + 1 pts (b) Correct eigenvectors/eigenspaces: $$ E_1 = \left\{ \left[\begin{array}{c}t \\ 0 \\ -t\end{array}\right]: t \in \mathbb{R}\right\} $$ $$ E_2 = \left\{ \left[\begin{array}{c}0 \\ t \\0 \end{array}\right]: t \in \mathbb{R}\right\} $$ $$ E_5 = \left\{ \left[\begin{array}{c}t \\ 0 \\t \end{array}\right]: t \in \mathbb{R}\right\} $$

✓ + 1 pts (b) Correct diagonalization: $$ A = SDS^{T} =SDS^{-1}$$ where $$ S = \left[\begin{array}{ccc} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{array} \right]$$ and $$D = \left[\begin{array}{ccc} 1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 5\end{array} \right]$$

✓ + 1 pts (c) $$q$$ is positive definite, or positive semidefinite (or correct answer based on eigenvalues from (b))

✓ + 2 pts (d) This level set is an ellipsoid (or correct shape based on computations from (b))

✓ + 2 pts (d) Rewrote quadratic form as $$q(\vec{x}) = c_1^2 + 2c_2^2 + 5c_3^2$$, or other justification using computations from (b)

+ 1 pts (a) (partial) Gave a matrix that either satisfies $$q(\vec{x}) = \vec{x} \cdot A\vec{x}$$, or is symmetric, but not both

QUESTION 7

7 Question 7 10 / 10

✓ + 10 pts Correct

Part a correct

+ **1.5 pts** T(1 2 3 4 5) = (1 2 3 4 5)
+ **1 pts** T e_1 = -e_1
+ **1 pts** T e_2 = 10 e_2
+ **1.5 pts** dim ker T = 2

+ **2 pts** state that diagonalization is the way to go
+ **3 pts** complete diagonalization
+ **10 pts** Did not simplify SDS^{-1}
+ **9 pts** small error in (b) in finishing up; ie, arithmetic or appropriate form
- **2 pts** 2 or more errors in (b), or more significant error (like putting columns of S in the wrong order, very incorrect characteristic polynomial)
+ **3 pts** Make clear that you see a pattern
+ **2 pts** correct statement of final answer. This deduction is deemed appropriately, because if you choose to do the problem by just looking for a pattern and not carrying out the diagonalization, then correctly discerning what exactly the pattern is much of the difficulty of the problem.
+ **0 pts** No work

✓ **+ 10 pts** Correct

QUESTION 8

8 Question 8 **10 / 10**

(a)

+ **2 pts** compute B^T B
+ **2 pts** get eigenvalues
+ **1 pts** square root

(b)

+ **1 pts** V matrix
+ **1 pts** Sigma matrix (with dimensions and placement matching U and V)
+ **1.5 pts** First two columns of U
+ **0.5 pts** U is square
+ **1 pts** third column of U (this point is not award if U is only given 2 columns, or if the third column is incorrect)
+ **2 pts** Awarded if reasonable effort is shown for each part of (b), but fewer than 2 other points in (b) are awarded. This is not awarded concurrently with the other points but in lieu of them.

Frank Zheng

Math 33A Final

Question 1

A. $x_1 + 2x_2 + 3x_3 - 4x_4 = 1$
$x_1 + 2x_2 + 3x_3 + 2x_4 = 2$
$2x_1 + 4x_2 + 6x_3 + 4x_4 = 4$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & -4 & 1 \\ 1 & 2 & 3 & 2 & 2 \\ 2 & 4 & 6 & 4 & 4 \end{array}\right] \xrightarrow{-1\times,\ -2\times} \left[\begin{array}{cccc|c} 1 & 2 & 3 & -4 & 1 \\ 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 12 & 2 \end{array}\right] \xrightarrow{+\frac{2}{3}\times,\ -2\times} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 5/3 \\ 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \times 1/6$$

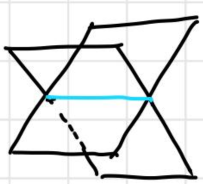
$$\rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 5/3 \\ 0 & 0 & 0 & 1 & 1/6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} x_1 + 2x_2 + 3x_3 = 5/3 \\ x_4 = 1/6 \end{array} \quad (x_2 = t,\ x_3 = s) \rightarrow \{(5/3 - 2t - 3s, t, s, 1/6) \mid t, s \in \mathbb{R}\}$$

B. $\text{rank}(A) = ?$

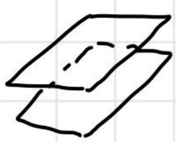
$$A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 1 & 2 & 3 & 2 \\ 2 & 4 & 6 & 4 \end{bmatrix}, \ \text{rref}(A) = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{rank}(A) = 2$$

The solution set represents a plane.

C. Geometrically, either a

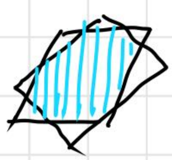


line or (∞ sols)



∅ or a (no sols)

planes are the same



plane (∞ sols)

1 Question 1 **10 / 10**

✓ **+ 5 pts** **a) Full credit: Convert the augmented system into rref form and get \[(1, 2, 3, 0); (0, 0, 0, 1); (0, 0, 0, 0)| 5/3; 1/6, 0]. Two free variables, x_2:=t, x_3:=s. Solution set is then {(5/3-2t-3s, t, s, 1/6)| s, t, real numbers}.**

✓ **+ 3 pts** **b) Full credit: The rank is 2 as there are 2 pivots. The solution set is a plane in R^4.**

✓ **+ 2 pts** **c) Full credit: The solution set of the combined system is the intersection of the two planes; it could possibly be empty (if the planes don't intersect) a point (for example, if the planes are in R^4 for instance and orthogonal complements of each other), a line, or a plane (if the two planes are the same).**

+ **3 pts** a) partial credit: convert to rref
+ **1 pts** a) partial credit: label free variables
+ **1 pts** a) partial credit: express solution set in terms of free variables (or alternatively find a basis)
+ **1.5 pts** b) partial credit: only one of rank=2 and plane given correctly
+ **1 pts** c) Partial credit: part of the solution, or some things correct and some incorrect
+ **0 pts** c) Mostly incorrect

Question 2

A. injective but not surjective

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}$

Cannot be surjective b/c $3 > 2$.

Injective b/c $\forall \vec{x}_1, \vec{x}_2 \in \mathbb{R}^2$ s.t. $\vec{x}_1 \neq \vec{x}_2$,

$T(\vec{x}_1) = (\vec{x}_1, 0) \neq (\vec{x}_2, 0) = T(\vec{x}_2)$

3-vector

B. surjective but not injective

$T: \mathbb{R}^2 \to \mathbb{R}$

$T(\vec{x}) = [1 \;\; 0]\vec{x}$

Surjective b/c $\forall y \in \mathbb{R}$, $f(y, 0) = y$.

Not injective b/c $f(1, 0) = f(1, 1) = 1$

C. $T: \mathbb{R}^m \to \mathbb{R}^n$ injective

$S: \mathbb{R}^n \to \mathbb{R}^p$ injective

$R = S \circ T$

$\forall \vec{x}_1, \vec{x}_2 \in \mathbb{R}^m$ where $\vec{x}_1 \neq \vec{x}_2$,

1. $T(\vec{x}_1) \neq T(\vec{x}_2)$ by injectivity of T b/c $\vec{x}_1 \neq \vec{x}_2$.
2. $S(T(\vec{x}_1)) \neq S(T(\vec{x}_2))$ by injectivity of S b/c $T(\vec{x}_1) \neq T(\vec{x}_2)$.

That is, $R(\vec{x}_1) = S(T(\vec{x}_1)) \neq S(T(\vec{x}_2)) = R(\vec{x}_2)$, satisfying the definition of injectivity.

Further, b/c T & S are linear, $\exists$ an $n \times m$ matrix A and a $p \times n$ matrix B s.t.

$T(\vec{x}) = A\vec{x}$ and $S(\vec{x}) = B\vec{x}$

$\to R(\vec{x}) = S(T(\vec{x})) = BA\vec{x}$

Since R is characterized by the $p \times m$ matrix BA, R is linear. ⊠

D. $T: \mathbb{R}^4 \to \mathbb{R}^4$ s.t. $T(1,2,3,4) = (2,2,3,4)$, $T(\vec{e}_2) = \vec{e}_3$, $T(\vec{e}_3) = \vec{e}_2$, $T(1,0,0,1) = \vec{e}_4$

1. $T(\vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3 + 4\vec{e}_4) = T(\vec{e}_1) + 2T(\vec{e}_2) + 3T(\vec{e}_3) + 4T(\vec{e}_4)$

$= T(\vec{e}_1) + 2\vec{e}_3 + 3\vec{e}_2 + 4T(\vec{e}_4) = (2, 2, 3, 4)$

$\rightarrow T(\vec{e}_1) + 4T(\vec{e}_4) = (2, -1, 1, 4)$

2. $T(\vec{e}_1 + \vec{e}_4) = T(\vec{e}_1) + T(\vec{e}_4) = \vec{e}_4$

$\rightarrow \vec{e}_4 + 3T(\vec{e}_4) = (2, -1, 1, 4)$

$\rightarrow T(\vec{e}_4) = (2/3, -1/3, 1/3, 1)$ $\rightarrow$ $\begin{bmatrix} -2/3 & 0 & 0 & 2/3 \\ 1/3 & 0 & 1 & -1/3 \\ -1/3 & 1 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\rightarrow T(\vec{e}_1) = (-2/3, 1/3, -1/3, 0)$

2 Question 2 **10 / 10**

✓ **+ 1 pts** **a) Injective but not surjective**

✓ **+ 1 pts** **b) surjective but not injective**

✓ **+ 3 pts** **c) Full credit: If x is a nonzero element of Rm then T injective implies T(x) is nonzero, and then S injective implies S(T(x)) is nonzero, and so R(x) is nonzero. Therefore R is injective, as its kernel contains only the zero vector. (Not needed as this was already shown in class, but we know R is a linear transformation since R(x+y)=S(T(x)+T(y))=S(T(x))+S(T(y))=R(x)+R(y) by the linearity of S and T, and also for any scalar c, R(c(x))=S(c(T(x))=c(S(T(x))=x(R(x)) again by the linearity of S and T.)**

✓ **+ 5 pts** **d) Full credit: Correct answer given by (-2/3, 0, 0, 2/3; 1/3, 0, 1, -1/3; -1/3, 1, 0, 1/3; 0, 0, 0, 1). Show work by either using change of basis matrix, or by computing 3T(e_4)=T(1, 2, 3, 4)-T(1, 0, 0, 1)-2T(e_2)-3T(e_3)=(2, 2, 3, 4)-e_4-2e_3-3e_2=(2, -1, 1, 3) which implies T(e_4)=(2/3, -1/3, 1/3, 1) and by computing T(e_1)=T(1, 0, 0, 1)-T(e_4)=e_4-(2/3, -1/3, 1/3, 1) =(-2/3, 1/3, -1/3, 0)**

+ 2 pts c) Partial credit: Nearly correct proof that is missing a minor justification

+ 1 pts c) Partial credit: A reasonable attempt that does not give a valid proof, but still makes use of the definition of injectivity of S and T to conclude something about the injectivity of R.

+ 3 pts d) Partial credit: Correct answer with some but insufficient justification

+ 0 pts 0 points

Question 3

V subspace $\mathbb{R}^n$

$P\vec{x} = \text{proj}_V(\vec{x})$

A. $\forall \vec{x} \in \mathbb{R}^n$,

$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ where $\vec{x}^{\parallel} \in V$ and $\vec{x}^{\parallel} \cdot \vec{x}^{\perp} = 0$ ↰

Suppose $\vec{v}$ is an eigenvector of P

$P\vec{v} = P(\vec{v}^{\parallel} + \vec{v}^{\perp}) = P\vec{v}^{\parallel} + P\vec{v}^{\perp}$

$= \text{proj}_V(\vec{v}^{\parallel}) + \text{proj}_V(\vec{v}^{\perp})$

$= \vec{v}^{\parallel} + 0$

* I use this defn. in other parts of this question.

Since $P\vec{v} = \lambda\vec{v}$ for $\lambda \in \mathbb{R}$,

$\vec{v}^{\parallel} = \lambda\vec{v} = \lambda\vec{v}^{\parallel} + \lambda\vec{v}^{\perp}$

The only way this is possible is if

1. $\vec{v}^{\parallel} = \vec{0}$, $\lambda = 0$
2. $\vec{v}^{\perp} = \vec{0}$, $\lambda = 1$ → $\lambda = 0, 1$

This makes sense geometrically as well; vectors perpendicular to V are collapsed to $\vec{0}$ ($\lambda = 0$) and vectors in V are unchanged ($\lambda = 1$) by proj_V.

B. $E_0 = \{\vec{v} \text{ s.t. } \vec{v}^{\parallel} = 0\}$ (from #1 in part A)

This defines $V^{\perp}$ since

($\Rightarrow$) If $\vec{v}^{\parallel} = 0$, $\forall \vec{x} \in V$, $\vec{v} \cdot \vec{x} = \vec{v}^{\parallel} \cdot \vec{x} + \underbrace{\vec{v}^{\perp} \cdot \vec{x}}_{0} = 0$, so $\vec{v} \in V^{\perp}$.

($\Leftarrow$) If $\vec{v} \in V^{\perp}$, $\vec{v} \cdot \vec{x} = 0$ (by definition). Thus,

$\vec{v}^{\parallel} \cdot \vec{x} + \underbrace{\vec{v}^{\perp} \cdot \vec{x}}_{0} = 0$

$\rightarrow \vec{v}^{\parallel} \cdot \vec{x} = 0 \rightarrow \vec{v}^{\parallel} = 0 \rightarrow \vec{v} \in E_0$.

Thus, $E_0 = V^{\perp}$.

$E_1 = \{\vec{v} \text{ s.t. } \vec{v}^{\perp} = 0\}$ (from #2 in part A)

That is, for $\vec{v} \in E_1$, $\vec{v} = \vec{v}^{\parallel} \in V$ (and if $\vec{v} \in V$, $\vec{v} \in E_1$ b/c $v^{\perp} = 0$), so $E_1 = V$.

C. Yes. We know $\dim(V) + \dim(V^{\perp}) = n$, so $\text{gemu}(E_0) + \text{gemu}(E_1) = n$, allowing us to construct an eigenbasis and diagonalize P.

D. Let $\vec{x} \in \mathbb{R}^n$. Then

$$P\vec{x} = P\vec{x}^{\parallel} + P\vec{x}^{\perp} = \vec{x}^{\parallel} + 0 = \vec{x}^{\parallel}$$

$$P^2\vec{x} = P(P\vec{x}) = P(\vec{x}^{\parallel} + 0) = P\vec{x}^{\parallel} = \vec{x}^{\parallel}$$

Thus, $P\vec{x} = P^2\vec{x}$, meaning P and P^2 represent the same transformation. Since transformations are uniquely categorized by a matrix, $P = P^2$. ⊠

3 Question 3 10 / 10

✓ + 1 pts (a): show 0,+1 are possible eigenvalues of P.

Solution:

For nonzero $$\vec{x} \in V$$ (assuming $$V \neq \{\vec{0}\}$$) we have $$P\vec{x} = \vec{x}$$ so $$x$$ has eigenvalue 1 for $$P$$; so 1 is an eigenvalue of $$P$$.

For nonzero $$\vec{x} \in V^{\perp}$$ (assuming $$V^{\perp} \neq \{\vec{0}\}$$) we have $$P\vec{x} = \vec{0}$$ so $$\vec{x}$$ has eigenvalue 0 for $$P$$; so $$0$$ is an eigenvalue for $$P$$.

✓ + 1 pts (a): show no eigenvalues other than 0,+1 are possible for P.

Solution:

Let $$\lambda$$ be an eigenvalue of $$P$$; let $$\vec{x}\neq\vec{0}$$ be an eigenvector of $$P$$ with eigenvalue $$\lambda$$; so $$\lambda\vec{x} = P\vec{x}$$. Next, due to the decomposition $$\mathbb{R}^n = V \oplus V^{\perp}$$ we can (uniquely) pick $$\vec{v} \in V$$ and $$\vec{v}_{\perp} \in V^{\perp}$$ such that $$\vec{x} = \vec{v} + \vec{v}_{\perp}$$; then now $$P(\vec{x}) = P(\vec{v} + \vec{v}_{\perp}) = \vec{v}$$.

So now $$\vec{v} = P\vec{x} = \lambda \vec{x} = \lambda \vec{v} + \lambda \vec{v}_{\perp}$$, so $$(1-\lambda)\vec{v} = \lambda \vec{v}_{\perp}$$. But now the left-hand-side $$(1-\lambda)\vec{v}$$ lies in $$V$$ while the right-hand-side $$\lambda \vec{v}_{\perp}$$ lies in $$V^{\perp}$$; so both sides are $$\vec{0}$$ (since $$V \cap V^{\perp} = \{\vec{0}\}$$).

So now both $$(1-\lambda)\vec{v} = \vec{0}$$ and $$\lambda \vec{v}_{\perp} = \vec{0}$$.

Case 1: if $$\vec{v}_{\perp} \neq \vec{0}$$, we must have $$\lambda = 0$$. Case 2: if $$\vec{v}_{\perp} = \vec{0}$$ (which then implies $$\vec{v}\neq\vec{0}$$ since $$\vec{x}\neq\vec{0}$$ and $$\vec{x} = \vec{v}+\vec{v}_{\perp}$$), then we must have $$1-\lambda=0$$.

So $$\lambda \in \{0,+1\}$$ are the only possibilities.

\[Alternate method 1: use geometric argument. If $$\vec{x}\neq\vec{0}$$ is an eigenvector of $$P$$ with eigenvalue not zero, then $$P\vec{x}$$ is both nonzero and parallel to $$\vec{x}$$. Since $$P$$ is an orthogonal projection, it is only possible for $$\vec{x},P\vec{x}$$ to be nonzero and parallel to each other if $$\vec{x} \in V$$.]

\[Alternate method 2: use the fact that $$P^2 = P$$ to argue eigenvalues must satisfy $$\lambda^2 = \lambda$$, whose only solutions are $$\lambda = 0,+1$$]

\[Alternate method 3: show that the 0-eigenspace is $$V$$ and the +1-eigenspace is $$V^{\perp}$$, so the sum of these eigenspaces has dimension $$n$$, and therefore is all of $$\mathbb{R}^n$$; therefore no eigenvalues other than 0,+1 are possible.

This proof uses repeatedly the fact that any two eigenspaces (for distinct eigenvalues) (of the same matrix) intersect trivially (meaning their intersection is $$\{\vec{0}\}$$), and so the dimension of their sum is the sum of their dimensions]

✓ + 2 pts (b): conclude 0-eigenspace is $$V^{\perp}$$ and +1-eigenspace is $$V$$.

Proof:

Let $$E_0$$ and $$E_1$$ denote the eigenspaces of $$P$$ for eigenvalues $$0$$ and $$+1$$ respectively.

First, already P acts as multiplication by 0 on $V^{\perp}$ and as multiplication by 1 on V. (Since if $\vec{v} \in V$ then $P\vec{v} = \vec{v}$, and if $\vec{v}^{\perp} \in V^{\perp}$ then $P\vec{v}^{\perp} = \vec{0}$.) This proves $V^{\perp} \subseteq E_0$ and $V \subseteq E_1$.

It only remains to show the converse: that eigenvectors of P with eigenvalue 0 are in $V^{\perp}$, and that eigenvectors of P with eigenvalue 1 are in V. (In other words, that $E_0 \subseteq V^{\perp}$ and that $E_1 \subseteq V$.)

\[Method 1]

Let $\vec{x} \in \mathbb{R}^n$ be a given eigenvector of P (with $\vec{x}\neq\vec{0}$) with eigenvalue λ; so $\lambda \vec{x} = P\vec{x}$. Using $\mathbb{R}^n = V \oplus V^{\perp}$ write $\vec{x} = \vec{v} + \vec{v}^{\perp}$ for some $\vec{v} \in V$ and some $\vec{v}^{\perp}\in V^{\perp}$.
From part (a) we must have $\lambda = 0$ or $\lambda = 1$.
We want to show that if $\lambda = 0$ then $\vec{x} \in V^{\perp}$, and that if $\lambda = 1$ then $\vec{x} \in V$.
Note $P\vec{x} = P(\vec{v}+\vec{v}^{\perp}) = \vec{v}$.
Case 1: if $\lambda = 0$, then $\vec{0} = 0\vec{x} = P\vec{x} = \vec{v}$; so $\vec{v}=\vec{0}$, therefore $\vec{x} = \vec{v}^{\perp} \in V^{\perp}$. Case 2: if $\lambda = 1$, then $\vec{x} = 1\vec{x} = P\vec{x} = \vec{v}$; so $\vec{x} = \vec{v} \in V$.

This completes the proof that $V=E_1$ and $V^{\perp} = E_0$.

\[Method 2]

We have shown already that $V^{\perp} \subseteq E_0$ and $V \subseteq E_1$. It follows that $V{\perp} + V \subseteq E_0 + E_1$. But $V^{\perp}+V = \mathbb{R}^n$; it must be the case that $E_0 + E_1 = \mathbb{R}^n$.

Next, we know $E_0 \cap E_1 = \{\vec{0}\}$ since E_0,E_1 are eigenspaces with distinct eigenvalues; it follows that $\mathrm{dim}(E_0+E_1) = \mathrm{dim}E_0+\mathrm{dim}E_1$. But $E_0 + E_1 = \mathbb{R}^n$. So now we have $n = \mathrm{dim}E_0+\mathrm{dim}E_1$.

Let $k := \mathrm{dim}V^{\perp}$; so $\mathrm{dim}V = n-k$ (because $V\oplus V^{\perp} = \mathbb{R}^n$). It follows that $\mathrm{dim}E_0 \geq k$ and $\mathrm{dim}E_1 \geq n-k$ (since $E_0 \supseteq V^{\perp}$ and $E_1 \supseteq V$).

Substituting $\mathrm{dim}E_1 = n-\mathrm{dim}E_0$ in the second relation above gives $n-\mathrm{dim}E_0\geq n-k$, which is equivalent to $\mathrm{dim}E_0 \leq k$.

So we have found $\mathrm{dim}E_0\geq k$ and $\mathrm{dim}E_0 \leq k$; it follows that $\mathrm{dim}E_0 = k$. And since $\mathrm{dim}E_1 +\mathrm{dim}E_0=n$ we then have $\mathrm{dim}E_1 = n-k$.

Finally, since $V^{\perp} \subseteq E_0$ and $\mathrm{dim}V^{\perp} = k = \mathrm{dim}E_0$ we must have $E_0 = V^{\perp}$;
similarly, since $V\subseteq E_1$ and $\mathrm{dim}V = n-k = \mathrm{dim}E_1$ we must have $E_1 = V$.
✓ + 1 pts (b): showing work/justification/proof
✓ + 1 pts (c): correctly say P can be diagonalized
✓ + 1 pts (c): provide sufficient justification for diagonalizability.
(e.g. the eigenspaces of P span all of $\mathbb{R}^n$, since $\mathbb{R}^n = V^{\perp} \oplus V = E_0^{(P)} \oplus E_{+1}^{(P)}$)
(equivalently: the geometric multiplicities of the eigenspaces add to n)
(more precisely: we can diagonalize $P = SDS^{-1}$ where the columns of S consist of a basis for V followed by a basis for $V^{\perp}$, and D is diagonal with first 1's for the V basis and 0's for the $V^{\perp}$ basis.)

\[alternate method: show P is symmetric, then apply spectral theorem]
✓ + 2 pts (d): argue that $P^2 = P$.

\[Method 1]

We recall P is diagonalizable (as shown in part (c)), and its eigenvalues are only $0,1$.

It follows that we can write $P = SDS^{-1}$ where S,D are square matrices (of the same size as P) with S invertible, and D diagonal with all diagonal entries of D being either 0 or 1.
In particular it follows that $D^2 = D$ (because D is diagonal, and each diagonal entry t of D is either 0 or 1 and therefore satisfies $t^2 = t$).

Now $P^2 = (SDS^{-1})^2 = (SDS^{-1})(SDS^{-1}) = SD(S^{-1}S)DS^{-1} = SDIDS^{-1} = S(D^2)S^{-1} = SDS^{-1} = P;$ note we used $D^2 = D$.

\[Method 2]

Recall that $P\vec{a} \in V$ for any vector $\vec{a} \in \mathbb{R}^n$. Furthermore, we also know $P\vec{v} = \vec{v}$ for any $\vec{v} \in V$.

It follows that for any vector $\vec{x} \in \mathbb{R}^n$ we have $P(P\vec{x}) = P\vec{x}$; so $P^2\vec{x} = P\vec{x}$.

Since P^2 and P agree on all vectors $\vec{x} \in \mathbb{R}^n$ it follows that $P^2 = P$.

✓ **+ 1 pts** (d): showing work/justification

Question 4

A. A: 2×2 45° CCW rotation
$\mathcal{B} = \{(1,1), (0,1)\}$

$A\vec{v}_1 = A(1,1) = (0, \sqrt{2}) = \sqrt{2}(0,1) \rightarrow [A\vec{v}_1]_{\mathcal{B}} = (0, \sqrt{2})$
$A\vec{v}_2 = (-\sqrt{2}/2, \sqrt{2}/2) = -\sqrt{2}/2\,(1,1) + \sqrt{2}(0,1) \rightarrow [A\vec{v}_2]_{\mathcal{B}} = (-\sqrt{2}/2, \sqrt{2})$
$$[A]_{\mathcal{B}} = \begin{bmatrix} [A\vec{v}_1]_{\mathcal{B}} & [A\vec{v}_2]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 0 & -\sqrt{2}/2 \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$$

b. $\mathcal{B}$ basis of $\mathbb{R}^n$

Define S for $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ as
$$S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

Then:
$$AB = S[AB]_{\mathcal{B}}S^{-1}$$
$$A = S[A]_{\mathcal{B}}S^{-1}$$
$$B = S[B]_{\mathcal{B}}S^{-1}$$
$$\rightarrow S[AB]_{\mathcal{B}}S^{-1} = AB = (S[A]_{\mathcal{B}}S^{-1})(S[B]_{\mathcal{B}}S^{-1})$$
$$= S([A]_{\mathcal{B}}[B]_{\mathcal{B}})S^{-1}$$
$$\rightarrow [AB]_{\mathcal{B}} = [A]_{\mathcal{B}}[B]_{\mathcal{B}}$$ ☒

c. A: 3×3 rotation
$\mathcal{B}$ basis of $\mathbb{R}^3$

Defining S as in part B,
$$A = S^{-1}[A]_{\mathcal{B}}S$$
$$\det(A) = \det(S^{-1})\det([A]_{\mathcal{B}})\det(S)$$
$$= \det([A]_{\mathcal{B}})$$

$$\det(I_n) = \det(SS^{-1})$$
$$= \det(S)\det(S^{-1}) = 1$$

Since A is a rotation matrix, it's columns must form a (rotated) cube in $\mathbb{R}^3$ w/ vol=1 that has not been reflected. Thus, $\det([A]_{\mathcal{B}}) = \det(A) = 1$

4 Question 4 10 / 10

✓ **+ 1 pts** (a): correct matrix $$A = \begin{bmatrix}
\cos45^{\circ} & -\sin45^{\circ}\\
\sin45^{\circ} & \cos45^{\circ}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}$$

✓ **+ 1 pts** (a): correct change-of-basis formula: $$[A]_{\mathcal{B}} = S^{-1}AS$$ where the columns of $$S$$ are the ordered basis vectors in $$\mathcal{B}$$
(or other method)

✓ **+ 1 pts** (a): correctly identify $$S = \begin{bmatrix}
1 & 0\\
1 & 1
\end{bmatrix}$$, and compute $$S^{-1} = \begin{bmatrix}
1 & 0\\
-1 & 1
\end{bmatrix}$$

+ 0 pts temp

✓ **+ 1 pts** (a): compute $$[A]_{\mathcal{B}} = \begin{bmatrix}
0 & -\frac{1}{\sqrt{2}}\\
\sqrt{2} & \sqrt{2}
\end{bmatrix}$$

✓ **+ 1 pts** (b): correct change-of-basis formulas for $$[A]_{\mathcal{B}}$$, $$[B]_{\mathcal{B}}$$, $$[AB]_{\mathcal{B}}$$

✓ **+ 2 pts** (b): directly compute:
$$[A]_{\mathcal{B}} [B]_{\mathcal{B}} = (S^{-1}AS)(S^{-1}BS) = S^{-1}A(SS^{-1})BS = S^{-1}AIBS = S^{-1}(AB)S = [AB]_{\mathcal{B}}.$$
Here $$S$$ is the matrix whose columns are the vectors in $$\mathcal{B}$$ (in the same order as in $$\mathcal{B}$$).

✓ **+ 1 pts** (c): argue that a rotation matrix has determinant $$\pm1$$.
For example, a geometric argument can be used, by applying the geometric definition of determinant. Rotations preserve lengths (as well as angles and volumes) so a rotation maps a cube to another cube of the same volume.

\[Specifically, the determinant is +1 for orientation-preserving rotations (known as proper rotations), and -1 for orientation-reversing rotations (known as improper rotations).]

\[Note: just +1 instead of ±1 will also be accepted, as the textbook defines a rotation as having determinant +1.]

Alternatively, we can argue rotation matrices are orthogonal matrices (because they preserve length), and orthogonal matrices must have determinant $$\pm1$$.

✓ + 2 pts (c): argue that $$A$$ and $$[A]_{\mathcal{B}}$$ have the same determinant.

Proof: $$\mathrm{det}([A]_{\mathcal{B}}) = \mathrm{det}(S^{-1}AS) = \mathrm{det}(S^{-1})\mathrm{det}(A)\mathrm{det}(S) = \mathrm{det}(A) \,\mathrm{det}(S)^{-1}\mathrm{det}(S) = \mathrm{det}(A).$$
Note here we used the multiplicative properties of determinant (i.e., $$\mathrm{det}(CC') = \mathrm{det}(C)\mathrm{det}(C')$$ for any square matrices $$C,C'$$ of the same size).
\[More generally, any two similar square matrices have the same determinant.]

Alternatively, we can argue that the determinant of a linear operator (or square matrix) can be calculated in a basis-independent way (i.e. the geometric definition of determinant, which looks at how the operator changes the volume of oriented parallelepipeds, etc), and so the determinant should be the same in any basis.

Question 5

A. $T: \mathbb{R}^2 \to \mathbb{R}^2$

$\{T(1,5), T(1,0)\} = \{(2,3), (4,5)\}$

$\det(A) = \text{Area}\{(1,5),(1,0)\} / \text{Area}\{(2,3),(4,5)\}$

$= \det\begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix} / \det\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$

$= -5/(10-12) = 5/2$

$\det(A^T A) = \det(A^T)\det(A) = (5/2)^2 = 25/4$

$\det((A^T A)^{-1}) = 1/\det(A^T A) = 4/25$

area of Q: $\sqrt{2}^2 = 2$

area of $(A^T A)^{-1} Q$: $\det((A^T A)^{-1}) \cdot \text{Area}(Q) = 4/25 \cdot 2 = 8/25$

B. $\det\begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 4 & 1 \end{bmatrix}$

A nonnull pattern must include:

- 4 in entry (4,3)
- 1 in entry (1,2) since row 4 is already used

Then:

$\det A = \text{sgn}(P_1)\,\text{prod}(P_1) + \text{sgn}(P_2)\,\text{prod}(P_2)$

$= (-1)^2 \cdot (1 \cdot 2 \cdot 1 \cdot 4) + (-1)^3 (1 \cdot 1 \cdot 1 \cdot 4)$

$= 8 - 4 = 4$

C. $\det\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{bmatrix}$ $-R_1$ $= \det\begin{bmatrix} 1 & 2 & \cdots & 6 & 7 \\ 1 & 1 & \cdots & 1 & 1 \\ 2 & 2 & \cdots & 2 & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 6 & 6 & \cdots & 6 & 6 \end{bmatrix}$

redundant rows, NOT invertible!

$= 0$

5 Question 5 **8 / 10**

✓ **+ 2 pts (a) Made a connection between areas and determinants**

+ **1 pts** (a) Computed $$A =\left[\begin{array}{cc} 4 & -2/5 \\ 5 & -2/5\end{array}\right]$$ or $$A =\left[\begin{array}{cc} 2 & 2/5 \\ 3 & 2/5\end{array}\right]$$, or its determinant of $$\pm 2/5$$

✓ **+ 1 pts (a) Used $$\det A = \det A^T$$ or $$\det A^{-1} = 1/\det A$$, or computed $$(A^TA)^{-1}$$ directly**

+ **1 pts** (a) Correct final answer: $$\displaystyle\frac{25}{2}$$

✓ **+ 1 pts (b) Correct determinant: $$4$$**

✓ **+ 1 pts (b) Showed work, or partial credit for work towards a determinant**

✓ **+ 1 pts (c) Correct determinant: $$0$$**

✓ **+ 2 pts (c) Showed work, or partial credit for work towards a determinant**

(1) This should be 2/5, not 5/2

Question 6

$q(x_1, x_2, x_3) = 3x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_3$

A. $A = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$

B. $\det(A - \lambda I_n) = (3-\lambda)(2-\lambda)(3-\lambda) - 4(2-\lambda)$

$= (2-\lambda)((3-\lambda)^2 - 4)$

$= (2-\lambda)(\lambda^2 - 6\lambda + 9 - 4)$

$= (2-\lambda)(\lambda - 5)(\lambda - 1) \rightarrow \lambda = 1, 2, 5$

$E_1 = \ker(A - I_n) = \ker \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \{(-t, 0, t) \mid t \in \mathbb{R}\} = \text{span}\{(-1, 0, 1)\}$

$E_2 = \ker(A - 2I_n) = \ker \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span}\{(0, 1, 0)\}$

$E_5 = \ker(A - 5I_n) = \ker \begin{bmatrix} -2 & 0 & 2 \\ 0 & -3 & 0 \\ 2 & 0 & -2 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span}\{(1, 0, 1)\}$

$\mathcal{B} = \{(-\sqrt{2}/2, 0, \sqrt{2}/2), (0, 1, 0), (\sqrt{2}/2, 0, \sqrt{2}/2)\}$ ($\lambda=1$, $\lambda=2$, $\lambda=5$)

$A = SBS^{-1} = \begin{bmatrix} -\sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -\sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix}$

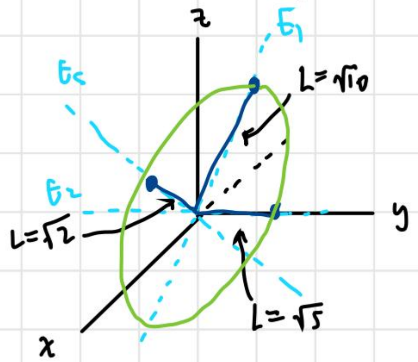
($S^{-1} = S^T$)

C. $\lambda = 1, 2, 5 > 0 \rightarrow$ positive definite

D. $\{\vec{x} \in \mathbb{R}^3 \mid q(\vec{x}) = 10\}$

$q(\vec{x}) = c_1^2 + 2c_2^2 + 5c_3^2 = 10$

That is, this level set describes the ellipsoid with the principal axes described by the eigenspaces of A. Something like:



Light blue = principal axes
Dark blue = ellipsoid "radii"
(definitely the wrong term)

6 Question 6 **10 / 10**

✓ **+ 2 pts** (a) Correct matrix: $$A = \left[\begin{array}{ccc} 3 & 0 & 2\\ 0 & 2 & 0\\ 2 & 0 & 3\end{array} \right]$$

✓ **+ 1 pts** (b) Correct eigenvalues: $$\lambda = 1,2,5$$

✓ **+ 1 pts** (b) Correct eigenvectors/eigenspaces: $$ E_1 = \left\{ \left[\begin{array}{c}t \\ 0 \\ -t\end{array}\right]: t \in \mathbb{R}\right\} $$ $$ E_2 = \left\{ \left[\begin{array}{c}0 \\ t \\0 \end{array}\right]: t \in \mathbb{R}\right\} $$ $$ E_5 = \left\{ \left[\begin{array}{c}t \\ 0 \\t \end{array}\right]: t \in \mathbb{R}\right\} $$

✓ **+ 1 pts** (b) Correct diagonalization: $$ A = SDS^{T} =SDS^{-1}$$ where $$ S = \left[\begin{array}{ccc} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{array} \right]$$ and $$D = \left[\begin{array}{ccc} 1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 5\end{array} \right]$$

✓ **+ 1 pts** (c) $$q$$ is positive definite, or positive semidefinite (or correct answer based on eigenvalues from (b))

✓ **+ 2 pts** (d) This level set is an ellipsoid (or correct shape based on computations from (b))

✓ **+ 2 pts** (d) Rewrote quadratic form as $$q(\vec{x}) = c_1^2 + 2c_2^2 + 5c_3^2$$, or other justification using computations from (b)

+ 1 pts (a) (partial) Gave a matrix that either satisfies $$q(\vec{x}) = \vec{x} \cdot A\vec{x}$$, or is symmetric, but not both

Question 7

A. $T: \mathbb{R}^5 \to \mathbb{R}^5$

i. $T((1,2,3,4,5)) = (1,2,3,4,5)$

ii. $T(\vec{e}_1) = -\vec{e}_1$

iii. $T(\vec{e}_2) = 10\vec{e}_2$

iv. $\dim(\ker(T)) = \dim(\ker(A)) = 2$

$\text{rank}(A) = \dim(\text{im}(A)) = 5-2 = 3.$

$T((1,2,3,4,5)) = T(\vec{e}_1) + 2T(\vec{e}_2) + 3T(\vec{e}_3) + 4T(\vec{e}_4) + 5T(\vec{e}_5)$

$= -\vec{e}_1 + 20\vec{e}_2 + 3T(\vec{e}_3) + 4T(\vec{e}_4) + 5T(\vec{e}_5)$

$\to 3T(\vec{e}_3) + 4T(\vec{e}_4) + 5T(\vec{e}_5) = (2,-18,3,4,5)$

To make things easy, let's try:

$T(\vec{e}_4) = \vec{0}$, $T(\vec{e}_5) = \vec{0}$ * guarantees $\text{rank}(A) \leq 3$

$\to 3T(\vec{e}_3) = (2,-18,3,4,5)$

$T(\vec{e}_3) = (2/3, -6, 1, 4/3, 5/3)$

$$A = \begin{bmatrix} -1 & 0 & 2/3 & 0 & 0 \\ 0 & 10 & -6 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4/3 & 0 & 0 \\ 0 & 0 & 5/3 & 0 & 0 \end{bmatrix}$$

* Check

i. $T(1,2,3,4,5) = (-1+2, 20-18, 3, 4, 5)$

$= (1,2,3,4,5)$ ✓

ii. $T(\vec{e}_1) = -\vec{e}_1$ ✓

iii. $T(\vec{e}_2) = 10\vec{e}_2$ ✓

iv. $\ker(A) = \ker\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} = \{(0,0,0,t,s) \mid t,s \in \mathbb{R}\}$ ✓

B. $$A - \lambda I_n = \begin{bmatrix} -1-\lambda & 0 & 2/3 & 0 & 0 \\ 0 & 10-\lambda & -6 & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 4/3 & -\lambda & 0 \\ 0 & 0 & 5/3 & 0 & -\lambda \end{bmatrix}$$

Only 1 nonnull pattern!

$\to \det(A-\lambda I_n) = (-1-\lambda)(10-\lambda)(1-\lambda)(-\lambda)^2$

$\to \lambda = -1, 0, 1, 10$

$$E_{-1} = \ker(A + I_n) = \ker\begin{bmatrix} 0 & 0 & 2/3 & 0 & 0 \\ 0 & -9 & -6 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 4/3 & 1 & 0 \\ 0 & 0 & 5/3 & 0 & 1 \end{bmatrix} \xrightarrow{\text{rref}} = \ker\begin{bmatrix} 0 & I_4 \\ 0 & 0 \end{bmatrix} = \{(t, \vec{0}) \mid t \in \mathbb{R}\} = \text{span}\{\vec{e}_1\}$$

$E_0 = \ker(A) = \ker(\text{rref}(A)) = \ker\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} = \{(0,0,0,t,s) \mid t,s \in \mathbb{R}\} = \text{span}\{\vec{e}_4, \vec{e}_5\}$

$E_1 = \ker(A - I_n) = \ker\begin{bmatrix} -2 & 0 & 2/3 & 0 & 0 \\ 0 & 9 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & 5/3 & 0 & -1 \end{bmatrix} = \ker\begin{bmatrix} -2 & 0 & 0 & 1/2 & 0 \\ 0 & 9 & 0 & -9/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & 0 & 5/4 & -1 \end{bmatrix}$

$= \ker\begin{bmatrix} 1 & 0 & 0 & 0 & -1/5 \\ 0 & 1 & 0 & 0 & -2/5 \\ 0 & 0 & 1 & 0 & -3/5 \\ 0 & 0 & 0 & 1 & -4/5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \{t(1/5, 2/5, 3/5, 4/5, 1) \mid t \in \mathbb{R}\} = \text{span}\{(1/5, 2/5, 3/5, 4/5, 1)\}$

$E_{10} = \ker(A - 10I_n) = \ker\begin{bmatrix} -11 & 0 & 2/3 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & -9 & 0 & 0 \\ 0 & 0 & 4/3 & -10 & 0 \\ 0 & 0 & 5/3 & 0 & -10 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \text{span}\{\vec{e}_2\}$

$\lambda = -1$, $\lambda = 10$, $\lambda = 0$, $\lambda = 1$

$\mathfrak{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_4, \vec{e}_5, (1,2,3,4,5)\}$

$S = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 5 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$S^{-1}: \left[\begin{array}{ccccc|ccccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{c|ccccc} & 1 & 0 & -1/3 & 0 & 0 \\ & 0 & 1 & -2/3 & 0 & 0 \\ I_5 & 0 & 0 & 1/3 & 0 & 0 \\ & 0 & 0 & -4/3 & 1 & 0 \\ & 0 & 0 & -5/3 & 0 & 1 \end{array}\right] \leftarrow S^{-1}$

$A^{2020} = (SBS^{-1})^{2020} = SB^{2020}S^{-1}$

$= \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} (-1)^{2020} & & & & 0 \\ & 10^{2020} & & & \\ & & 1^{2020} & & \\ & & & & \\ 0 & & & & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1/3 & 0 & 0 \\ 0 & 1 & -2/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & -4/3 & 1 & 0 \\ 0 & 0 & -5/3 & 0 & 1 \end{bmatrix}$

$$= S \begin{bmatrix} 1 & 0 & -1/3 & 0 & 0 \\ 0 & 10^{2020} & -2/3 \cdot 10^{2020} & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 10^{2020} & -2/3 \cdot 10^{2020} + 2/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4/3 & 0 & 0 \\ 0 & 0 & 5/3 & 0 & 0 \end{bmatrix}}$$

7 Question 7 **10 / 10**

✓ **+ 10 pts** Correct

Part a correct

+ 1.5 pts T(1 2 3 4 5) = (1 2 3 4 5)

+ 1 pts T e_1 = -e_1

+ 1 pts T e_2 = 10 e_2

+ 1.5 pts dim ker T = 2

+ 2 pts state that diagonalization is the way to go

+ 3 pts complete diagonalization

+ 10 pts Did not simplify SDS^{-1}

+ 9 pts small error in (b) in finishing up; ie, arithmetic or appropriate form

- 2 pts 2 or more errors in (b), or more significant error (like putting columns of S in the wrong order, very incorrect characteristic polynomial)

+ 3 pts Make clear that you see a pattern

+ 2 pts correct statement of final answer. This deduction is deemed appropriately, because if you choose to do the problem by just looking for a pattern and not carrying out the diagonalization, then correctly discerning what exactly the pattern is much of the difficulty of the problem.

+ 0 pts No work

Question 8

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

A. $B^T B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$\det(B^T B - \lambda I_n) = (3-\lambda)(2-\lambda) \rightarrow \lambda = 2, 3,$ $\sigma_1 = \sqrt{3}, \ \sigma_2 = \sqrt{2}$

B. $E_2 = \ker(B^T B - 2I_n) = \ker \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \text{span}\{(0,1)\} \leftarrow \vec{v}_2$

$E_3 = \ker(B^T B - 3I_n) = \ker \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = \text{span}\{(1,0)\} \leftarrow \vec{v}_1$

$L(\vec{v}_1) = B\vec{v}_1 = B\vec{e}_1 = (1, 1, 1)$

$L(\vec{v}_2) = B\vec{v}_2 = B\vec{e}_2 = (1, -1, 0)$

$V = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}, \ U = \begin{bmatrix} | & | & | \\ L(\vec{v}_1)/\sigma_1 & L(\vec{v}_2)/\sigma_2 & ? \\ | & | & | \end{bmatrix}$

$\vec{u}_1 = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$

$\vec{u}_2 = (1/\sqrt{2}, -1/\sqrt{2}, 0)$

$\text{proj}_{\{\vec{u}_1, \vec{u}_2\}}(\vec{e}_3) = 1/\sqrt{3}\,\vec{u}_1 = (1/3, 1/3, 1/3)$

$\vec{u}_3 = \dfrac{\vec{e}_3 - \text{proj}_{\{\vec{u}_1, \vec{u}_2\}}(\vec{e}_3)}{\|\vec{e}_3 - \text{proj}_{\{\vec{u}_1, \vec{u}_2\}}(\vec{e}_3)\|} = \dfrac{(-1/3, -1/3, 2/3)}{1/3\sqrt{6}} = (-1/\sqrt{6}, -1/\sqrt{6}, 2/\sqrt{6})$

$\rightarrow U \Sigma V^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

8 Question 8 **10 / 10**

(a)

+ **2 pts** compute B^T B
+ **2 pts** get eigenvalues
+ **1 pts** square root

(b)

+ **1 pts** V matrix
+ **1 pts** Sigma matrix (with dimensions and placement matching U and V)
+ **1.5 pts** First two columns of U
+ **0.5 pts** U is square
+ **1 pts** third column of U (this point is not award if U is only given 2 columns, or if the third column is incorrect)
+ **2 pts** Awarded if reasonable effort is shown for each part of (b), but fewer than 2 other points in (b) are awarded. This is not awarded concurrently with the other points but in lieu of them.

✓ + **10 pts** **Correct**