1. (a) (5 points) Find the set of solutions of the linear system

$$x_1 + 2x_2 + 3x_3 - 4x_4 = 1$$
$$x_1 + 2x_2 + 3x_3 + 2x_4 = 2$$
$$2x_1 + 4x_2 + 6x_3 + 4x_4 = 4$$

(b) (3 points) Write the above linear system in matrix form $A\vec{x} = \vec{b}$ and compute rank(A). Describe the solution set geometrically.

(c) (2 points) Suppose that the solution set to one given linear system of equations is a plane, and the solution set to another given linear system of equations is also a plane. What are the possible solution sets of the system of linear equations obtained by combining the equations in both systems?

a)
$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & -4 & 1 \\ 1 & 2 & 3 & 2 & 2 \\ 2 & 4 & 6 & 4 & 4 \end{array}\right] \xrightarrow{R2-R1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & -4 & 1 \\ 0 & 0 & 0 & 6 & 1 \\ 2 & 4 & 6 & 4 & 4 \end{array}\right]$$

$$\xrightarrow{R3-2R2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & -4 & 1 \\ 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 12 & 2 \end{array}\right] \xrightarrow{R3-2R2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & -4 & 1 \\ 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R1+\frac{2}{3}R2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 5/3 \\ 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_2, x_3 are free

$x_2 = s \quad x_3 = t$

$x_1 + 2x_2 + 3x_3 = 5/3 \qquad x_1 = \frac{5}{3} - 2x_2 - 3x_3 = \frac{5}{3} - 2s - 3t$

$6x_4 = 1 \quad x_4 = \frac{1}{6}$

$$\boxed{\left\{ \left(5/3 - 2s - 3t, s, t, \tfrac{1}{6} \right) \mid s, t \in \mathbb{R} \right\}}$$

b)
$$\boxed{\begin{bmatrix} 1 & 2 & 3 & -4 \\ 1 & 2 & 3 & 2 \\ 2 & 4 & 6 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}}$$

$\boxed{\text{rank}(A) = 2 \text{ since there are 2 pivots}}$

$\boxed{\text{The solution set is a plane in } \mathbb{R}^4}$

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c)

1. No solutions when the planes never intersect
2. A solution set of points along the line where the two planes intersect.
3. Infinitely many solutions when the two planes intersect/overlap everywhere

1 Question 1 **10 / 10**

✓ **+ 5 pts** **a) Full credit: Convert the augmented system into rref form and get \[(1, 2, 3, 0); (0, 0, 0, 1); (0, 0, 0, 0)| 5/3; 1/6, 0]. Two free variables, x_2:=t, x_3:=s. Solution set is then {(5/3-2t-3s, t, s, 1/6)| s, t, real numbers}.**

✓ **+ 3 pts** **b) Full credit: The rank is 2 as there are 2 pivots. The solution set is a plane in R^4.**

✓ **+ 2 pts** **c) Full credit: The solution set of the combined system is the intersection of the two planes; it could possibly be empty (if the planes don't intersect) a point (for example, if the planes are in R^4 for instance and orthogonal complements of each other), a line, or a plane (if the two planes are the same).**

+ **3 pts** a) partial credit: convert to rref
+ **1 pts** a) partial credit: label free variables
+ **1 pts** a) partial credit: express solution set in terms of free variables (or alternatively find a basis)
+ **1.5 pts** b) partial credit: only one of rank=2 and plane given correctly
+ **1 pts** c) Partial credit: part of the solution, or some things correct and some incorrect
+ **0 pts** c) Mostly incorrect

2. (a) (1 point) Give an example of a linear transformation that is injective but not surjective.

(b) (1 point) Give an example of a linear transformation that is surjective but not injective.

(c) (3 points) Suppose that T is an injective linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ and that S is an injective linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^p$. Prove that composition $R = S \circ T$ defined by $R(\vec{x}) = S(T(\vec{x}))$ is an injective linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^p$.

(d) (5 points) Find the matrix for the linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$ that maps $(1, 2, 3, 4)$ to $(2, 2, 3, 4)$, maps e_2 to e_3, maps e_3 to e_2, and maps $(1, 0, 0, 1)$ to e_4.

a) A linear transformation that maps vectors in $\mathbb{R}^2$ to unique vectors in $\mathbb{R}^3$ is an example of a linear transformation that is injective but not surjective.

ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ maps $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to $\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$, thus the image is not all of $\mathbb{R}^3$ and A is not surjective. A is injective because no two vectors in $\mathbb{R}^2$ are mapped to the same vector in $\mathbb{R}^3$

b) A linear transformation that maps vectors in $\mathbb{R}^3$ to vectors in $\mathbb{R}^2$ would be surjective but not injective. This is because all of $\mathbb{R}^2$ can be mapped by transforming vectors in $\mathbb{R}^3$ and some vectors in $\mathbb{R}^3$ can be mapped to the same vector in $\mathbb{R}^2$.

ex: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ maps $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, thus the image is all of $\mathbb{R}^2$ and unique vectors in $\mathbb{R}^3$ can be mapped to ~~that~~ same vector in $\mathbb{R}^2$.

c) First, since T is injective, every unique vector $\vec{x}$ in $\mathbb{R}^m$ is mapped to a unique vector in $\mathbb{R}^n$. Since S is injective, all of those vectors in $\mathbb{R}^n$ are then mapped to unique vectors in $\mathbb{R}^p$. Additionally, if A and B are the transformation matrices for T and S, respectively, then $R(\vec{x}) = BA\vec{x}$. Since a transformation is injective if and only if it has a kernel $= \{\vec{0}\}$, we know that A and B have kernel $= \{\vec{0}\}$. cont ↓

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To find $\ker(BA)$, we say $BA\vec{x} = B\vec{v}$. Since $\ker(B) = \vec{0}$, we know that $\vec{v} = \vec{0}$ for $B\vec{v} = \vec{0}$. We then need to find all the vectors $\vec{x}$ so that $A\vec{x} = \vec{0}$. Since A is injective, only $\vec{x} = \vec{0}$ when $A\vec{x} = \vec{0}$. This means $\vec{x} = \vec{0}$ when $BA\vec{x} = \vec{0}$, meaning that $\ker(BA) = \{\vec{0}\}$ and that $R(\vec{x}) = S(T(\vec{x}))$ is an injective linear transformation from $\mathbb{R}^m \to \mathbb{R}^p$.

d)
$$-3\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} - 2\begin{bmatrix}0\\1\\0\\0\end{bmatrix} - 3\begin{bmatrix}0\\0\\1\\0\end{bmatrix} - 4\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

$$A\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = -\frac{1}{3}\left[\begin{bmatrix}2\\2\\3\\4\end{bmatrix} - 2\begin{bmatrix}0\\0\\1\\0\end{bmatrix} - 3\begin{bmatrix}0\\1\\0\\0\end{bmatrix} - 4\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right] = -\frac{1}{3}\begin{bmatrix}2\\-1\\1\\0\end{bmatrix}$$

$$3\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} - 2\begin{bmatrix}0\\1\\0\\0\end{bmatrix} - 3\begin{bmatrix}0\\0\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

$$A\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \frac{1}{3}\left[\begin{bmatrix}2\\2\\3\\4\end{bmatrix} - 2\begin{bmatrix}0\\0\\1\\0\end{bmatrix} - 3\begin{bmatrix}0\\1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right] = \frac{1}{3}\begin{bmatrix}2\\-1\\1\\3\end{bmatrix}$$

$$\boxed{A = \begin{bmatrix}-2/3 & 0 & 0 & 2/3\\ 1/3 & 0 & 1 & -1/3\\ -1/3 & 1 & 0 & 1/3\\ 0 & 0 & 0 & 1\end{bmatrix}}$$

2 Question 2 **10 / 10**

✓ **+ 1 pts** **a) Injective but not surjective**

✓ **+ 1 pts** **b) surjective but not injective**

✓ **+ 3 pts** **c) Full credit: If x is a nonzero element of Rm then T injective implies T(x) is nonzero, and then S injective implies S(T(x)) is nonzero, and so R(x) is nonzero. Therefore R is injective, as its kernel contains only the zero vector. (Not needed as this was already shown in class, but we know R is a linear transformation since R(x+y)=S(T(x)+T(y))=S(T(x))+S(T(y))=R(x)+R(y) by the linearity of S and T, and also for any scalar c, R(c(x))=S(c(T(x))=c(S(T(x))=x(R(x)) again by the linearity of S and T.)**

✓ **+ 5 pts** **d) Full credit: Correct answer given by (-2/3, 0, 0, 2/3; 1/3, 0, 1, -1/3; -1/3, 1, 0, 1/3; 0, 0, 0, 1). Show work by either using change of basis matrix, or by computing 3T(e_4)=T(1, 2, 3, 4)-T(1, 0, 0, 1)-2T(e_2)-3T(e_3)=(2, 2, 3, 4)-e_4-2e_3-3e_2=(2, -1, 1, 3) which implies T(e_4)=(2/3, -1/3, 1/3, 1) and by computing T(e_1)=T(1, 0, 0, 1)-T(e_4)=e_4-(2/3, -1/3, 1/3, 1) =(-2/3, 1/3, -1/3, 0)**

+ 2 pts c) Partial credit: Nearly correct proof that is missing a minor justification

+ 1 pts c) Partial credit: A reasonable attempt that does not give a valid proof, but still makes use of the definition of injectivity of S and T to conclude something about the injectivity of R.

+ 3 pts d) Partial credit: Correct answer with some but insufficient justification

+ 0 pts 0 points

3. Let V be a subspace of $\mathbb{R}^n$, and let P be the orthogonal projection matrix onto V. That is, P satisfies

$$P\vec{x} = \text{proj}_V \vec{x}, \qquad \text{for all } \vec{x} \text{ in } \mathbb{R}^n.$$

   (a) (2 points) What are the eigenvalues of P? Justify your answer.
   (b) (3 points) What are the corresponding eigenspaces? Justify your answer.
   (c) (2 points) Can P be diagonalized? Justify your answer.
   (d) (3 points) Prove that $P^2 = P$.

a) $\lambda = 1, 0$ because only two eigenspaces have vectors where $P\vec{x} = \lambda\vec{x}$. These two eigenspaces are V and $V^\perp$. Any vector on V will be mapped to itself by the projection, which is why $\lambda = 1$. Any vector on $V^\perp$ will be mapped to $\vec{0}$ by the projection, which is why $\lambda = 0$. Additionally, for projections, $P\vec{x} = P^2\vec{x}$ and if $P\vec{x} = \lambda\vec{x}$, then $\lambda\vec{x} = \lambda^2\vec{x}$ so $\lambda = \lambda^2$ meaning $\lambda = 0, 1$.

b) As mentioned earlier, $E_0 = V^\perp$ and $E_1 = V$. $E_0 = V^\perp$ because all vectors in $V^\perp$ will be mapped to $\vec{0}$ after the projection onto V. This corresponds to $P\vec{x} = \vec{0} = (0)\vec{x}$. $E_1 = V$ because all vectors in V will be mapped to themselves after the projection, which corresponds to $P\vec{x} = \vec{x} = (1)\vec{x}$.

c) Yes, P can be diagonalized. $V^\perp = \ker(P)$ for a projection matrix. $V = \text{im}(P)$ as well. By rank-nullity theorem, we know that $\dim(\ker(P)) + \dim(\text{im}(P)) = n$. This means that $\dim(V^\perp) + \dim(V) = n$. Furthermore, it means that we can construct a basis for $\mathbb{R}^n$ using the basis of $V^\perp$ and basis V. Since the basis of $V^\perp$ and the basis of V both consists of eigenvectors of P, we can say that A basis for $\mathbb{R}^n$ can be constructed with n eigenvectors of P. This proves that P is diagonizable.

d) Since P is diagonizable, $P^t = SB^tS^{-1}$ where $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
So for P^2, $B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ which means that $P^2 = S\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}S^{-1}$.
This shows $P^2 = P$ as $P = S\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}S^{-1}$ which means that $P^2 = P = S\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}S^{-1}$. Geometrically, $P^2\vec{x} = PP\vec{x}$ so P^2 is essentially projecting a vector $P\vec{x}$ that already lies in V, which as mentioned before, means that $P\vec{x}$ is mapped to itself after the projection.

3 Question 3 9 / 10

✓ + 1 pts (a): show 0,+1 are possible eigenvalues of P.

Solution:

For nonzero $$\vec{x} \in V$$ (assuming $$V \neq \{\vec{0}\}$$) we have $$P\vec{x} = \vec{x}$$ so $$x$$ has eigenvalue 1 for $$P$$; so 1 is an eigenvalue of $$P$$.

For nonzero $$\vec{x} \in V^{\perp}$$ (assuming $$V^{\perp} \neq \{\vec{0}\}$$) we have $$P\vec{x} = \vec{0}$$ so $$\vec{x}$$ has eigenvalue 0 for $$P$$; so $$0$$ is an eigenvalue for $$P$$.

✓ + 1 pts (a): show no eigenvalues other than 0,+1 are possible for P.

Solution:

Let $$\lambda$$ be an eigenvalue of $$P$$; let $$\vec{x}\neq\vec{0}$$ be an eigenvector of $$P$$ with eigenvalue $$\lambda$$; so $$\lambda\vec{x} = P\vec{x}$$. Next, due to the decomposition $$\mathbb{R}^n = V \oplus V^{\perp}$$ we can (uniquely) pick $$\vec{v} \in V$$ and $$\vec{v}_{\perp} \in V^{\perp}$$ such that $$\vec{x} = \vec{v} + \vec{v}_{\perp}$$; then now $$P(\vec{x}) = P(\vec{v} + \vec{v}_{\perp}) = \vec{v}$$.

So now $$\vec{v} = P\vec{x} = \lambda \vec{x} = \lambda \vec{v} + \lambda \vec{v}_{\perp}$$, so $$(1-\lambda)\vec{v} = \lambda \vec{v}_{\perp}$$. But now the left-hand-side $$(1-\lambda)\vec{v}$$ lies in $$V$$ while the right-hand-side $$\lambda \vec{v}_{\perp}$$ lies in $$V^{\perp}$$; so both sides are $$\vec{0}$$ (since $$V \cap V^{\perp} = \{\vec{0}\}$$).

So now both $$(1-\lambda)\vec{v} = \vec{0}$$ and $$\lambda \vec{v}_{\perp} = \vec{0}$$.

Case 1: if $$\vec{v}_{\perp} \neq \vec{0}$$, we must have $$\lambda = 0$$. Case 2: if $$\vec{v}_{\perp} = \vec{0}$$ (which then implies $$\vec{v}\neq\vec{0}$$ since $$\vec{x}\neq\vec{0}$$ and $$\vec{x} = \vec{v}+\vec{v}_{\perp}$$), then we must have $$1-\lambda=0$$.

So $$\lambda \in \{0,+1\}$$ are the only possibilities.

\[Alternate method 1: use geometric argument. If $$\vec{x}\neq\vec{0}$$ is an eigenvector of $$P$$ with eigenvalue not zero, then $$P\vec{x}$$ is both nonzero and parallel to $$\vec{x}$$. Since $$P$$ is an orthogonal projection, it is only possible for $$\vec{x},P\vec{x}$$ to be nonzero and parallel to each other if $$\vec{x} \in V$$.]

\[Alternate method 2: use the fact that $$P^2 = P$$ to argue eigenvalues must satisfy $$\lambda^2 = \lambda$$, whose only solutions are $$\lambda = 0,+1$$]

\[Alternate method 3: show that the 0-eigenspace is $$V$$ and the +1-eigenspace is $$V^{\perp}$$, so the sum of these eigenspaces has dimension $$n$$, and therefore is all of $$\mathbb{R}^n$$; therefore no eigenvalues other than 0,+1 are possible.

This proof uses repeatedly the fact that any two eigenspaces (for distinct eigenvalues) (of the same matrix) intersect trivially (meaning their intersection is $$\{\vec{0}\}$$), and so the dimension of their sum is the sum of their dimensions]

✓ + 2 pts (b): conclude 0-eigenspace is $$V^{\perp}$$ and +1-eigenspace is $$V$$.

Proof:

Let $$E_0$$ and $$E_1$$ denote the eigenspaces of $$P$$ for eigenvalues $$0$$ and $$+1$$ respectively.

First, already P acts as multiplication by 0 on $V^{\perp}$ and as multiplication by 1 on V. (Since if $\vec{v} \in V$ then $P\vec{v} = \vec{v}$, and if $\vec{v}^{\perp} \in V^{\perp}$ then $P\vec{v}^{\perp} = \vec{0}$.) This proves $V^{\perp} \subseteq E_0$ and $V \subseteq E_1$.

It only remains to show the converse: that eigenvectors of P with eigenvalue 0 are in $V^{\perp}$, and that eigenvectors of P with eigenvalue 1 are in V. (In other words, that $E_0 \subseteq V^{\perp}$ and that $E_1 \subseteq V$.)

[Method 1]

Let $\vec{x} \in \mathbb{R}^n$ be a given eigenvector of P (with $\vec{x}\neq\vec{0}$) with eigenvalue λ; so $\lambda \vec{x} = P\vec{x}$. Using $\mathbb{R}^n = V \oplus V^{\perp}$ write $\vec{x} = \vec{v} + \vec{v}^{\perp}$ for some $\vec{v} \in V$ and some $\vec{v}^{\perp}\in V^{\perp}$.
From part (a) we must have $\lambda = 0$ or $\lambda = 1$.
We want to show that if $\lambda = 0$ then $\vec{x} \in V^{\perp}$, and that if $\lambda = 1$ then $\vec{x} \in V$.
Note $P\vec{x} = P(\vec{v}+\vec{v}^{\perp}) = \vec{v}$.
Case 1: if $\lambda = 0$, then $\vec{0} = 0\vec{x} = P\vec{x} = \vec{v}$; so $\vec{v}=\vec{0}$, therefore $\vec{x} = \vec{v}^{\perp} \in V^{\perp}$. Case 2: if $\lambda = 1$, then $\vec{x} = 1\vec{x} = P\vec{x} = \vec{v}$; so $\vec{x} = \vec{v} \in V$.

This completes the proof that $V=E_1$ and $V^{\perp} = E_0$.

[Method 2]

We have shown already that $V^{\perp} \subseteq E_0$ and $V \subseteq E_1$. It follows that $V{\perp} + V \subseteq E_0 + E_1$. But $V^{\perp}+V = \mathbb{R}^n$; it must be the case that $E_0 + E_1 = \mathbb{R}^n$.

Next, we know $E_0 \cap E_1 = \{\vec{0}\}$ since E_0,E_1 are eigenspaces with distinct eigenvalues; it follows that $\mathrm{dim}(E_0+E_1) = \mathrm{dim}E_0+\mathrm{dim}E_1$. But $E_0 + E_1 = \mathbb{R}^n$. So now we have $n = \mathrm{dim}E_0+\mathrm{dim}E_1$.

Let $k := \mathrm{dim}V^{\perp}$; so $\mathrm{dim}V = n-k$ (because $V\oplus V^{\perp} = \mathbb{R}^n$). It follows that $\mathrm{dim}E_0 \geq k$ and $\mathrm{dim}E_1 \geq n-k$ (since $E_0 \supseteq V^{\perp}$ and $E_1 \supseteq V$).

Substituting $\mathrm{dim}E_1 = n-\mathrm{dim}E_0$ in the second relation above gives $n-\mathrm{dim}E_0\geq n-k$, which is equivalent to $\mathrm{dim}E_0 \leq k$.

So we have found $$\mathrm{dim}E_0\geq k$$ and $$\mathrm{dim}E_0 \leq k$$; it follows that $$\mathrm{dim}E_0 = k$$. And since $$ \mathrm{dim}E_1 +\mathrm{dim}E_0=n$$ we then have $$\mathrm{dim}E_1 = n-k$$.

Finally, since $$V^{\perp} \subseteq E_0$$ and $$\mathrm{dim}V^{\perp} = k = \mathrm{dim}E_0$$ we must have $$E_0 = V^{\perp}$$;
similarly, since $$V\subseteq E_1$$ and $$\mathrm{dim}V = n-k = \mathrm{dim}E_1$$ we must have $$E_1 = V$$.

+ **1 pts** (b): showing work/justification/proof

✓ + **1 pts** (c): correctly say P can be diagonalized

✓ + **1 pts** (c): provide sufficient justification for diagonalizability.

(e.g. the eigenspaces of $$P$$ span all of $$\mathbb{R}^n$$, since $$\mathbb{R}^n = V^{\perp} \oplus V = E_0^{(P)} \oplus E_{+1}^{(P)} $$)
(equivalently: the geometric multiplicities of the eigenspaces add to n)
(more precisely: we can diagonalize $$P = SDS^{-1}$$ where the columns of $$S$$ consist of a basis for $$V$$ followed by a basis for $$V^{\perp}$$, and $$D$$ is diagonal with first 1's for the $$V$$ basis and 0's for the $$V^{\perp}$$ basis.)

\[alternate method: show P is symmetric, then apply spectral theorem]

✓ + **2 pts** (d): argue that $$P^2 = P$$.

\[Method 1]

We recall $$P$$ is diagonalizable (as shown in part (c)), and its eigenvalues are only $$0,1$$.

It follows that we can write $$P = SDS^{-1}$$ where $$S,D$$ are square matrices (of the same size as $$P$$) with $$S$$ invertible, and $$D$$ diagonal with all diagonal entries of $$D$$ being either 0 or 1.
In particular it follows that $$D^2 = D$$ (because $$D$$ is diagonal, and each diagonal entry $$t$$ of $$D$$ is either 0 or 1 and therefore satisfies $$t^2 = t$$).

Now $$P^2 = (SDS^{-1})^2 = (SDS^{-1})(SDS^{-1}) = SD(S^{-1}S)DS^{-1} = SDIDS^{-1} = S(D^2)S^{-1} = SDS^{-1} = P;$$ note we used $$D^2 = D$$.

\[Method 2]

Recall that $$P\vec{a} \in V$$ for any vector $$\vec{a} \in \mathbb{R}^n$$. Furthermore, we also know $$P\vec{v} = \vec{v}$$ for any $$\vec{v} \in V$$.

It follows that for any vector $$\vec{x} \in \mathbb{R}^n$$ we have $$P(P\vec{x}) = P\vec{x}$$; so $$P^2\vec{x} = P\vec{x}$$.

Since $$P^2$$ and $$P$$ agree on all vectors $$\vec{x} \in \mathbb{R}^n$$ it follows that $$P^2 = P$$.

✓ + 1 pts (d): showing work/justification

4. (a) (4 points) Let A be the 2×2 matrix of rotation by 45 degrees counterclockwise in the plane. Let $[A]_{\mathcal{B}}$ denote the $\mathcal{B}$-matrix of A with respect to the basis $\{(1,1),(0,1)\}$. Find $[A]_{\mathcal{B}}$.

(b) (3 points) Let $\mathcal{B}$ be a basis of $\mathbb{R}^n$. Prove that

$$[AB]_{\mathcal{B}} = [A]_{\mathcal{B}}[B]_{\mathcal{B}}.$$

That is, prove that the $\mathcal{B}$-matrix of AB is equal to the $\mathcal{B}$-matrix of A times the $\mathcal{B}$-matrix of B.

(c) (3 points) Suppose that A is a rotation matrix in $\mathbb{R}^3$, and suppose that $\mathcal{B}$ is some basis of $\mathbb{R}^3$. What is the determinant of the $\mathcal{B}$-matrix of A? Justify your answer.

a) $A = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$[A]_{\mathcal{B}} = S^{-1}AS = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$= \boxed{\begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \end{bmatrix}}$

b) $[AB]_{\mathcal{B}} = S^{-1}ABS$

$[A]_{\mathcal{B}} = S^{-1}AS$

$[B]_{\mathcal{B}} = S^{-1}BS$

$SS^{-1} = I_n$

$[A]_{\mathcal{B}}[B]_{\mathcal{B}} = S^{-1}ASS^{-1}BS$

$= S^{-1}AI_nBS$

$= S^{-1}ABS$

$\boxed{[AB]_{\mathcal{B}} = [A]_{\mathcal{B}}[B]_{\mathcal{B}}}$ ✓

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c) $\mathcal{B}$-matrix $= S^{-1}AS$

$\det(S^{-1}AS) = \det(S^{-1})\det(A)\det(S)$

We know that $\det(S^{-1}) = \frac{1}{\det(S)}$

Additionally, a property of rotation matrices is that their determinant is 1. This means $\det(A) = 1$

As a result:

$$\det(S^{-1}AS) = \frac{1}{\det(S)}\det(A)\det(S)$$

$$= \frac{1}{\det(S)}(1)\det(S)$$

$$= \frac{\det(S)}{\det(S)} = 1$$

$\boxed{\det(\mathcal{B}\text{-matrix of } A) = 1}$

4 Question 4 **9 / 10**

✓ **+ 1 pts** (a): correct matrix $$A = \begin{bmatrix}
\cos45^{\circ} & -\sin45^{\circ}\\
\sin45^{\circ} & \cos45^{\circ}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}$$

✓ **+ 1 pts** (a): correct change-of-basis formula: $$[A]_{\mathcal{B}} = S^{-1}AS$$ where the columns of $$S$$ are the ordered basis vectors in $$\mathcal{B}$$
(or other method)

✓ **+ 1 pts** (a): correctly identify $$S = \begin{bmatrix}
1 & 0\\
1 & 1
\end{bmatrix}$$, and compute $$S^{-1} = \begin{bmatrix}
1 & 0\\
-1 & 1
\end{bmatrix}$$

+ 0 pts temp

✓ **+ 1 pts** (a): compute $$[A]_{\mathcal{B}} = \begin{bmatrix}
0 & -\frac{1}{\sqrt{2}}\\
\sqrt{2} & \sqrt{2}
\end{bmatrix}$$

✓ **+ 1 pts** (b): correct change-of-basis formulas for $$[A]_{\mathcal{B}}$$, $$[B]_{\mathcal{B}}$$, $$[AB]_{\mathcal{B}}$$

✓ **+ 2 pts** (b): directly compute:
$$[A]_{\mathcal{B}} [B]_{\mathcal{B}} = (S^{-1}AS)(S^{-1}BS) = S^{-1}A(SS^{-1})BS = S^{-1}AIBS = S^{-1}(AB)S = [AB]_{\mathcal{B}}.$$
Here $$S$$ is the matrix whose columns are the vectors in $$\mathcal{B}$$ (in the same order as in $$\mathcal{B}$$).

+ 1 pts (c): argue that a rotation matrix has determinant $$\pm1$$.
For example, a geometric argument can be used, by applying the geometric definition of determinant. Rotations preserve lengths (as well as angles and volumes) so a rotation maps a cube to another cube of the same volume.

\[Specifically, the determinant is +1 for orientation-preserving rotations (known as proper rotations), and -1 for orientation-reversing rotations (known as improper rotations).]

\[Note: just +1 instead of ±1 will also be accepted, as the textbook defines a rotation as having determinant +1.]

Alternatively, we can argue rotation matrices are orthogonal matrices (because they preserve length), and orthogonal matrices must have determinant $$\pm1$$.

✓ + 2 pts (c): argue that $$A$$ and $$[A]_{\mathcal{B}}$$ have the same determinant.

Proof: $$\mathrm{det}([A]_{\mathcal{B}}) = \mathrm{det}(S^{-1}AS) = \mathrm{det}(S^{-1})\mathrm{det}(A)\mathrm{det}(S) = \mathrm{det}(A) \,\mathrm{det}(S)^{-1}\mathrm{det}(S) = \mathrm{det}(A).$$
Note here we used the multiplicative properties of determinant (i.e., $$\mathrm{det}(CC') = \mathrm{det}(C)\mathrm{det}(C')$$ for any square matrices $$C,C'$$ of the same size).
\[More generally, any two similar square matrices have the same determinant.]

Alternatively, we can argue that the determinant of a linear operator (or square matrix) can be calculated in a basis-independent way (i.e. the geometric definition of determinant, which looks at how the operator changes the volume of oriented parallelepipeds, etc), and so the determinant should be the same in any basis.

1 need justification for why rotation matrix has determinant 1

5. (a) (5 points) Suppose that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix A maps the parallelogram formed by the vectors

$$\{(1,5),(1,0)\}$$

to the parallelogram formed by the vectors

$$\{(2,3),(4,5)\}.$$

If Q is a square of sidelength $\sqrt{2}$, what is the area of the image of Q under the linear transformation $(A^TA)^{-1}$? (That is, what is the area of $(A^TA)^{-1}(Q)$?)

(b) (2 points) Find the determinant of the 4×4 matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 4 & 1 \end{pmatrix}$$

Be sure to show your work. An answer without the relevant work shown will receive very little credit.

(c) (3 points) Find the determinant of the 7×7 matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{pmatrix}$$

Be sure to show your work. An answer without the relevant work shown will receive very little credit.

a) $\det\left[(A^TA)^{-1}(Q)\right] = \det(A^TA)^{-1}\det(Q)$

$= \left[\det(A^T)\det(A)\right]^{-1}\det(Q)$

$= \left[\det(A)^2\right]^{-1}\det(Q)$

$5\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix}$ $\quad A\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{5}\left[\begin{bmatrix}2\\3\end{bmatrix} - \begin{bmatrix}4\\5\end{bmatrix}\right] = \frac{1}{5}\begin{bmatrix}-2\\-2\end{bmatrix}$

$A = \begin{bmatrix}4 & -2/5\\5 & -2/5\end{bmatrix}$ $\quad \det(A) = -\frac{8}{5} + \frac{10}{5} = \frac{2}{5}$

$\det(Q) = (\sqrt{2})^2 = 2$

$\frac{2}{\left(\frac{2}{5}\right)^2} = \frac{2}{4/25} = \frac{\cancel{2}(25)}{\cancel{4}\,2} = \boxed{\frac{25}{2}}$

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b) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 4 & 1 \end{bmatrix} \xrightarrow{R2-R3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[\substack{R1-R2 \\ R4-R2}]{R3-R2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

$\xrightarrow{R4-R1-R3} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \div 4 \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2 swaps, divide row by 4

$\det(I) = 1$ $\det = 1 \cdot (-1)^2 \cdot (4) = \boxed{4}$

c) We can subtract the first row from all subsequent rows to get a matrix where each row except the first has the same value across its columns and all the rows increase from 1 to 6

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 \end{bmatrix}$$

We can then use the second row to zero out all following rows.

We can then subtract the second row from the first row. Then, we can swap the first row with the second to get an upper triangular matrix. Since the determinant of an uppertriangular matrix is equal to the product of all values on the diagonal, we know the determinant is 0 as we have 0's on the diagonal.

$\boxed{\det = 0}$

5 Question 5 **10 / 10**

- ✓ **+ 2 pts** (a) Made a connection between areas and determinants
- ✓ **+ 1 pts** (a) Computed $$A =\left[\begin{array}{cc} 4 & -2/5 \\ 5 & -2/5\end{array}\right]$$ or $$A =\left[\begin{array}{cc} 2 & 2/5 \\ 3 & 2/5\end{array}\right]$$, or its determinant of $$\pm 2/5$$
- ✓ **+ 1 pts** (a) Used $$\det A = \det A^T$$ or $$\det A^{-1} = 1/\det A$$, or computed $$(A^TA)^{-1}$$ directly
- ✓ **+ 1 pts** (a) Correct final answer: $$\displaystyle\frac{25}{2}$$
- ✓ **+ 1 pts** (b) Correct determinant: $$4$$
- ✓ **+ 1 pts** (b) Showed work, or partial credit for work towards a determinant
- ✓ **+ 1 pts** (c) Correct determinant: $$0$$
- ✓ **+ 2 pts** (c) Showed work, or partial credit for work towards a determinant

6. Define a quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$ by

$$q(x_1, x_2, x_3) = 3x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_3$$

(a) (2 points) Find the associated 3×3 symmetric matrix A such that $q(\vec{x}) = \vec{x} \cdot (A\vec{x})$.

(b) (3 points) Find an orthogonal diagonalization of A.

(c) (1 point) What is the definiteness of q?

(d) (4 points) Use the information from part(b) to geometrically describe the level set

$$\{\vec{x} \in \mathbb{R}^3 : q(\vec{x}) = 10\},$$

read as "the set of all vectors $\vec{x}$ in $\mathbb{R}^3$ such that $q(\vec{x}) = 10$." Be sure to explain how you use the information from part(b). You can sketch the level set if you'd like but it is not required.

a) $\boxed{A = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}}$

b) $\det \begin{bmatrix} 3-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & 3-\lambda \end{bmatrix} = (3-\lambda)^2(2-\lambda) - 4(2-\lambda) = 0$

$(2-\lambda)((3-\lambda)^2 - 4) = 0$

$(\lambda-2)[\lambda^2 - 6\lambda + 9 - 4] = 0 \qquad (\lambda-2)(\lambda-5)(\lambda-1) = 0$

$\lambda_1 = 5 \quad \lambda_2 = 2 \quad \lambda_3 = 1$

$E_5 = \ker \begin{bmatrix} -2 & 0 & 2 \\ 0 & -3 & 0 \\ 2 & 0 & -2 \end{bmatrix} \qquad \begin{bmatrix} -2 & 0 & 2 \\ 0 & -3 & 0 \\ 2 & 0 & -2 \end{bmatrix} \xrightarrow{R3+R1} \begin{bmatrix} -2 & 0 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$-2x_1 + 2x_3 = 0 \qquad -3x_2 = 0 \qquad x_3$ free

$x_1 = x_3 \quad x_2 = 0 \quad x_3 = 1$

$E_5 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

$E_2 = \ker \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{R3-2R1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \div -3 \xrightarrow{R1-2R3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$x_1 = 0 \quad x_3 = 0 \quad x_2$ free, $x_2 = 1$

$E_2 = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

cont

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$$E_1 = \ker \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} \xrightarrow{R3-R1} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_3 \quad x_2 = 0 \quad x_3$ free, $x_3 = 1$

$$E_1 = \text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$E_1 \text{ ONB} = \left\{ \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

$$E_5 \text{ ONB} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\} \quad E_2 \text{ ONB} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$A = QBQ^{-1} \quad \boxed{Q = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}} \quad \boxed{B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Q^{-1} on next page →

c) $\boxed{q \text{ is positive definite as all its eigenvalues } > 0}$

d) we can write $q(\vec{x}) = 10$ as $\lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 = 10$, which, using the eigenvalues found in part b, gives us

$q(\vec{x}) = 5c_1^2 + 2c_2^2 + c_3^2 = 10$

This is an ellipsoid with axes along the principle axes:

$$\underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\text{derived from eigenspaces in part b}}. \quad c_1 = \sqrt{2} \quad c_2 = \sqrt{5} \quad c_3 = \sqrt{10}$$

The half-length of each ellipsoid axis, as measured from the ellipsoid center to the surface of the ellipsoid, is:

ellipsoid axis along:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{length} = \sqrt{2}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{length} = \sqrt{5}, \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{length} = \sqrt{10}$$

$$\left[\begin{array}{ccc|ccc} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 & 0 & 1 \end{array}\right] \xrightarrow{R3 - R1} \left[\begin{array}{ccc|ccc} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \div 1/\sqrt{2} \\ \\ \div \frac{2}{\sqrt{2}} \end{array}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{array}\right] \xrightarrow{R1+R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{array}\right]$$

$$\boxed{Q^{-1} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}}$$

6 Question 6 **10 / 10**

✓ **+ 2 pts** **(a) Correct matrix: $$A = \left[\begin{array}{ccc} 3 & 0 & 2\\ 0 & 2 & 0\\ 2 & 0 & 3\end{array} \right]$$**

✓ **+ 1 pts** **(b) Correct eigenvalues: $$\lambda = 1,2,5$$**

✓ **+ 1 pts** **(b) Correct eigenvectors/eigenspaces: $$ E_1 = \left\{ \left[\begin{array}{c}t \\ 0 \\ -t\end{array}\right]: t \in \mathbb{R}\right\} $$ $$ E_2 = \left\{ \left[\begin{array}{c}0 \\ t \\0 \end{array}\right]: t \in \mathbb{R}\right\} $$ $$ E_5 = \left\{ \left[\begin{array}{c}t \\ 0 \\t \end{array}\right]: t \in \mathbb{R}\right\} $$**

✓ **+ 1 pts** **(b) Correct diagonalization: $$ A = SDS^{T} =SDS^{-1}$$ where $$ S = \left[\begin{array}{ccc} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{array} \right]$$ and $$D = \left[\begin{array}{ccc} 1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 5\end{array} \right]$$**

✓ **+ 1 pts** **(c) $$q$$ is positive definite, or positive semidefinite (or correct answer based on eigenvalues from (b))**

✓ **+ 2 pts** **(d) This level set is an ellipsoid (or correct shape based on computations from (b))**

✓ **+ 2 pts** **(d) Rewrote quadratic form as $$q(\vec{x}) = c_1^2 + 2c_2^2 + 5c_3^2$$, or other justification using computations from (b)**

+ 1 pts (a) (partial) Gave a matrix that either satisfies $$q(\vec{x}) = \vec{x} \cdot A\vec{x}$$, or is symmetric, but not both

7. (a) (5 points) Find a $5x5$ matrix A whose corresponding linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$ satisfies **all** of the following four criteria, $(i), (ii), (iii), (iv)$:

   (i) $T((1,2,3,4,5)) = (1,2,3,4,5)$,
   (ii) $T(\vec{e_1}) = -\vec{e_1}$,
   (iii) $T(\vec{e_2}) = 10\vec{e_2}$,
   (iv) $\dim(\ker(T)) = \dim(\ker(A)) = 2$, i.e., the dimension of the kernel of A is 2.

   (b) (5 points) Compute A^{2020}, being sure to show your work.

a) $T((1,2,3,4,5)) = (1,2,3,4,5)$ $\lambda = 1$

$T(\vec{e}_1) = -\vec{e}_1$ $\lambda = -1$

$T(\vec{e}_2) = 10\vec{e}_2$ $\lambda = 10$

$E_1 = \text{span}\left\{\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}\right\}$ $E_{-1} = \text{span}\left\{\begin{bmatrix}-1\\0\\0\\0\\0\end{bmatrix}\right\}$ $E_{10} = \left\{\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}\right\}$

Assume additional eigenvalue $\lambda = 0$ alg. mult $= 2$

Assume $E_0 = \text{span}\{\vec{e_3}, \vec{e_4}\}$

$$S = \begin{bmatrix}1&0&0&0&1\\0&1&0&0&2\\0&0&1&0&3\\0&0&0&1&4\\0&0&0&0&5\end{bmatrix} \qquad \left[\begin{array}{ccccc|ccccc}1&0&0&0&1&1&0&0&0&0\\0&1&0&0&2&0&1&0&0&0\\0&0&1&0&3&0&0&1&0&0\\0&0&0&1&4&0&0&0&1&0\\0&0&0&0&5&0&0&0&0&1\end{array}\right] \div 5$$

$$\xrightarrow[\substack{R3-3R5\\R4-4R5}]{\substack{R1-R5\\R2-2R5}} \left[\begin{array}{ccccc|ccccc}1&0&0&0&0&1&0&0&0&-1/5\\0&1&0&0&0&0&1&0&0&-2/5\\0&0&1&0&0&0&0&1&0&-3/5\\0&0&0&1&0&0&0&0&1&-4/5\\0&0&0&0&1&0&0&0&0&1/5\end{array}\right] \qquad S^{-1} = \begin{bmatrix}1&0&0&0&-1/5\\0&1&0&0&-2/5\\0&0&1&0&-3/5\\0&0&0&1&-4/5\\0&0&0&0&1/5\end{bmatrix}$$

$$B = \begin{bmatrix}-1&0&0&0&0\\0&10&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{bmatrix} \qquad A = SBS^{-1} = \boxed{\begin{bmatrix}-1&0&0&0&2/5\\0&10&0&0&-18/5\\0&0&0&0&3/5\\0&0&0&0&4/5\\0&0&0&0&1\end{bmatrix}}$$

$$SB = \begin{bmatrix}-1&0&0&0&1\\0&10&0&0&2\\0&0&0&0&3\\0&0&0&0&4\\0&0&0&0&5\end{bmatrix}$$

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b) $A^t = SB^tS^{-1}$

$$B^t = \begin{bmatrix} (-1)^t & 0 & 0 & 0 & 0 \\ 0 & (10)^t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^{2020} = SB^{2020}S^{-1}$$

$$B^{2020} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 10^{2020} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$SB^{2020} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 10^{2020} & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$SB^{2020}S^{-1} = \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 10^{2020} & 0 & 0 & \frac{2(1-10^{2020})}{5} \\ 0 & 0 & 0 & 0 & 3/5 \\ 0 & 0 & 0 & 0 & 4/5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}$$

7 Question 7 **10 / 10**

✓ **+ 10 pts** **Correct**

Part a correct

+ 1.5 pts T(1 2 3 4 5) = (1 2 3 4 5)

+ 1 pts T e_1 = -e_1

+ 1 pts T e_2 = 10 e_2

+ 1.5 pts dim ker T = 2

+ 2 pts state that diagonalization is the way to go

+ 3 pts complete diagonalization

+ 10 pts Did not simplify SDS^{-1}

+ 9 pts small error in (b) in finishing up; ie, arithmetic or appropriate form

- 2 pts 2 or more errors in (b), or more significant error (like putting columns of S in the wrong order, very incorrect characteristic polynomial)

+ 3 pts Make clear that you see a pattern

+ 2 pts correct statement of final answer. This deduction is deemed appropriately, because if you choose to do the problem by just looking for a pattern and not carrying out the diagonalization, then correctly discerning what exactly the pattern is much of the difficulty of the problem.

+ 0 pts No work

8. Let

$$B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$$

(a) (5 points) Find the singular values of the matrix B given above. Show all your steps. An answer given without the relevant work will receive very little credit.

(b) (5 points) Find a singular value decomposition $B = U\Sigma V^T$ of the matrix B. Show all your steps. An answer given without the relevant work will receive very little credit.

a) $B^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

$B^T B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$\det \begin{bmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} = (3-\lambda)(2-\lambda) = 0 \quad \lambda_1 = 3 \quad \lambda_2 = 2$

$\boxed{\sigma_1 = \sqrt{3} \quad \sigma_2 = \sqrt{2}}$

b) $E_3 = \ker \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ $x_2 = 0$ x_1 free; $x_1 = 1$

$= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$E_2 = \ker \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $x_1 = 0$ x_2 free, $x_2 = 1$

$= \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$B\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $B\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\boxed{V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$ $\boxed{U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}}$ $\boxed{\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}}$

8 Question 8 **10 / 10**

(a)

+ **2 pts** compute B^T B
+ **2 pts** get eigenvalues
+ **1 pts** square root

(b)

+ **1 pts** V matrix
+ **1 pts** Sigma matrix (with dimensions and placement matching U and V)
+ **1.5 pts** First two columns of U
+ **0.5 pts** U is square
+ **1 pts** third column of U (this point is not award if U is only given 2 columns, or if the third column is incorrect)
+ **2 pts** Awarded if reasonable effort is shown for each part of (b), but fewer than 2 other points in (b) are awarded. This is not awarded concurrently with the other points but in lieu of them.

✓ + **10 pts Correct**