Math 33 - 2nd midterm – friday march 3, 2017

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Discussion session group : 3E?? 2E

3+4+4+3+3+3
+1

Fill in your answers directly on this form. You may use both sides of the paper. Please make sure you always clearly justify your answers and detail your calculus.

Total 6 questions with 3 + 4 + 4+ 3 + 3 + 3 = 20 pts.

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Question 1. (3pt)

Find a basis for $W^{\perp}$ where $W = span\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}5\\6\\7\\8\end{bmatrix}\right)$.

seek a vector orthogonal to both

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\\4\end{pmatrix} = 0 \qquad \begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 = 0 \end{cases}$$

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\cdot\begin{pmatrix}5\\6\\7\\8\end{pmatrix} = 0$$

$$\left[\begin{array}{cccc|c}1&2&3&4&0\\5&6&7&8&0\end{array}\right] - 5\times(\mathrm{I}) = \left[\begin{array}{cccc|c}1&2&3&4&0\\0&-4&-8&-12&0\end{array}\right] \div -4 = \left[\begin{array}{cccc|c}1&2&3&4&0\\0&1&2&3&0\end{array}\right] - 2\times(\mathrm{II})$$

$$= \left[\begin{array}{cccc|c}1&0&-1&-2&0\\0&1&2&3&0\end{array}\right] \quad \begin{matrix} x_1 = x_3 + 2x_4 \\ x_2 = -2x_3 - 3x_4 \end{matrix}$$

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}t+2s\\-2t-3s\\t\\s\end{pmatrix} = \begin{pmatrix}1\\-2\\1\\0\end{pmatrix}t + \begin{pmatrix}2\\-3\\0\\1\end{pmatrix}s$$

Basis for $W^{\perp}$: $\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}2\\-3\\0\\1\end{bmatrix}$

As expected, $\dim(W) + \dim(W^{\perp}) = 2+2 = \underset{\mathbb{R}^4}{4}$

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Question 2. (4pt)
Find scalars a, b, c, d, e, f, g such that the vectors

$$\begin{bmatrix} a \\ d \\ f \end{bmatrix}, \begin{bmatrix} b \\ 1 \\ g \end{bmatrix}, \begin{bmatrix} c \\ e \\ 1/2 \end{bmatrix}$$

are orthonormal.

dot product is 0 length is 1

$$\begin{cases} ab + d + fg = 0 \\ ac + de + \frac{1}{2}f = 0 \\ bc + e + \frac{1}{2}g = 0 \end{cases} \qquad \begin{cases} a^2 + d^2 + f^2 = 1 \\ b^2 + g^2 = 0 \to b, g = 0 \\ e^2 + c^2 = \frac{3}{4} \end{cases}$$

$$\begin{bmatrix} a \\ 0 \\ f \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} c \\ e \\ \frac{1}{2} \end{bmatrix}$$

dot dot

$0 + g + 0 = 0$

$d = 0$ $e = 0$

$$\boxed{c^2 + 0^2 + \frac{1}{2}^2 = 1 \quad c^2 = \frac{3}{4}}$$

$c = \frac{\sqrt{3}}{2}$

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~~Let A: $\begin{bmatrix} a & b & c \\ d & 1 & e \\ f & g & \frac{1}{2} \end{bmatrix}$ If the vectors are orthonormal, A is an orthogonal matrix. det(A) = ±1~~

$\cancel{\det(A)} = a\left(\frac{1}{2} - eg\right) + b\left(ef - \frac{1}{2}d\right) + c\left(dg - f\right) = 1$

~~$b, g = 0$~~

~~$\frac{1}{2}a - fc = 1$~~

~~$a = 2(1 + fc)$~~

~~$\frac{1}{2}a - \frac{\sqrt{3}}{2}f = 1$~~

~~$a = 2 + \sqrt{3}f$~~

~~$\frac{1}{2}(2 + \sqrt{3}f) - \frac{\sqrt{3}}{2}f = 1$~~

$$\begin{bmatrix} a \\ 0 \\ f \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

$\frac{\sqrt{3}}{2}a + 0 + \frac{1}{2}f = 0$

$a^2 + f^2 = 1$

$f = -\sqrt{3}a$

$$\begin{cases} d = 0 \\ \cancel{ac + \frac{1}{2}f = 0} \\ e = 0 \end{cases}$$

$c^2 = \frac{3}{4}$

$c = \frac{\sqrt{3}}{2}$

$$\boxed{\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \\ 0 \\ 0 \\ -\frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}}$$

with vectors

$$\begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

$a^2 = \frac{1}{4}$

$a = \pm\frac{1}{2}$

$\frac{1}{4} + f^2 = 1$

$f^2 = \frac{3}{4}$

$f = \pm\frac{\sqrt{3}}{2}$

If we pick

$$\begin{cases} a = \frac{1}{2}, \text{ then } f = -\frac{\sqrt{3}}{2} \\ a = -\frac{1}{2}, \text{ then } f = \frac{\sqrt{3}}{2} \end{cases}$$

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Question 3. (4pt)

Find an orthonormal basis $\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}$ of $\mathbb{R}^3$ such that

$1+4+9$

$$span(\overrightarrow{u_1}) = span\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) \text{ and } span(\overrightarrow{u_1}, \overrightarrow{u_2}) = span\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}\right).$$

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$$l_1 = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

$$\vec{u}_1 = \frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{14}}\\\frac{2}{\sqrt{14}}\\\frac{3}{\sqrt{14}}\end{bmatrix}$$

Note $\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}1\\1\\-1\end{bmatrix} = 0$, they are orthogonal

$$l_2 = \sqrt{1^2+1^2+(-1)^2} = \sqrt{3}$$

$$\vec{u}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\end{bmatrix}$$

$\|\vec{u}_1 \times \vec{u}_2\| =$

$\vec{u}_3 = \vec{u}_1 \times \vec{u}_2$ $\quad \|\vec{u}_1\|\|\vec{u}_2\| \sin 90° = 1$ (so it will be a unit vector)

$$\begin{bmatrix}\frac{1}{\sqrt{14}}\\\frac{2}{\sqrt{14}}\\\frac{3}{\sqrt{14}}\end{bmatrix} \times \begin{bmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\end{bmatrix} = \det\begin{bmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}}\end{bmatrix} = \begin{bmatrix}-\frac{5}{\sqrt{42}}\\\frac{4}{\sqrt{42}}\\-\frac{1}{\sqrt{42}}\end{bmatrix}$$

$$l_3 = \sqrt{\left(-\frac{5}{\sqrt{42}}\right)^2 + \left(\frac{4}{\sqrt{42}}\right)^2 + \left(-\frac{1}{\sqrt{42}}\right)^2} = 1 \checkmark$$

$$\boxed{\vec{u}_1, \vec{u}_2, \vec{u}_3 = \begin{bmatrix}\frac{1}{\sqrt{14}}\\\frac{2}{\sqrt{14}}\\\frac{3}{\sqrt{14}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\end{bmatrix}, \begin{bmatrix}-\frac{5}{\sqrt{42}}\\\frac{4}{\sqrt{42}}\\-\frac{1}{\sqrt{42}}\end{bmatrix}}$$

3+1

Question 4. (3pt)

Find the least-squares solution $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$.

Also, calculate the error $\|\vec{b} - A\vec{x}^*\|$.

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$$

$$= \left(\begin{bmatrix} 3 & 5 & 4 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 3 & 5 & 4 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$$

15 + 45 + 8

10 + 27 + 10

$$= \begin{bmatrix} 50 & 41 \\ 41 & 38 \end{bmatrix}^{-1} \begin{bmatrix} 68 \\ 47 \end{bmatrix}$$

$$= \frac{1}{1900-1681} \begin{bmatrix} 38 & -41 \\ -41 & 50 \end{bmatrix} \begin{bmatrix} 68 \\ 47 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{38}{219} & -\frac{41}{219} \\ -\frac{41}{219} & \frac{50}{219} \end{bmatrix} \begin{bmatrix} 68 \\ 47 \end{bmatrix}$$

$41 \cdot 41 = 1681$; $1900 - 1681 = 219$

$68 \cdot 38 = 2584$; $2584 - 1927 = 657$

$41 \cdot 47 = 1927$

$68 \cdot 41 = 2788$; $2788 - 2350 = 438$

$47 \cdot 50 = 2350$; $2350 - 1927 = 423$

$\frac{657}{219}$, $\frac{423}{219}$

$$\boxed{\vec{x}^* = \begin{bmatrix} 3 \\ -2 \end{bmatrix}}$$

☹ y u do this to me

$$A\vec{x}^* = \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$$

Check

$$\left[\begin{array}{cc|c} 3 & 2 & 5 \\ 5 & 3 & 9 \\ 4 & 5 & 2 \end{array}\right] \rightarrow \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} x_2 = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$$

$$\|\vec{b} - A\vec{x}^*\| = \left\| \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix} - \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix} \right\| = 0$$

✓

$\vec{b}$ is in the image of A....

aha! good job

$$= \begin{bmatrix} 9 \\ 15 \\ 12 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$$

Question 5. (3pt)

Find the determinant of the following matrix $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{bmatrix}$. $4+2+1+1$

One pattern with non-zero contribution.

8 inversions, $sgn(P) = +$ $prod(P) = 1 \times 2 \times 3 \times 4 \times 5 = 120$

$$\boxed{det(A) = \sum sgn(P)\, prod(P) = 120}$$

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Question 6. (3pt)

Use Gaussian elimination to find the determinant of the following matrix
$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 4 & 4 \\ 1 & 1 & 1 & 1 & 5 \end{bmatrix} \end{array} \begin{array}{l} \\ -(\mathrm{I}) \\ -(\mathrm{I}) \\ -(\mathrm{I}) \\ -(\mathrm{I}) \end{array}.$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

subtracting multiples of another row, det(A) doesn't change.

Upper triangular.

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$$\boxed{\det(A) = 1 \times 1 \times 2 \times 3 \times 4 = 24}$$