MATH 33A - SECTION 2
MIDTERM #1

APRIL 18, 2014

Full Name	Evan Krause
Student ID	804 291 181
Discussion Section	2E

Problem 1	20 /20
Problem 2	30/30
Problem 3	30 /30
Problem 4	20 /20
Total	100/100

Problem 1. Solve the system of linear equations in the variables $x_1, \ldots, x_6$.

$$\left.\begin{aligned} x_1 + 2x_2 + 2x_3 + x_4 + x_5 &= 4 \\ x_1 + 2x_2 + x_5 - x_6 &= 0 \\ x_1 + 2x_2 + 2x_3 - x_5 + x_6 &= 2 \end{aligned}\right\}$$

Augmented Matrix:

$$\left[\begin{array}{cccccc|c} 1 & 2 & 2 & 1 & 1 & 0 & 4 \\ 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 1 & 2 & 2 & 0 & -1 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cccccc|c} 1 & 2 & 2 & 1 & 1 & 0 & 4 \\ 0 & 0 & -2 & -1 & 0 & -1 & -4 \\ 0 & 0 & 0 & -1 & -2 & 1 & -2 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{cccccc|c} 1 & 2 & 2 & 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 2 \\ 0 & 0 & 0 & 1 & 2 & -1 & 2 \end{array}\right] \begin{array}{l} \\ R_2 \to -\frac{1}{2}R_2 \\ R_3 \to -R_3 \end{array} \rightarrow \left[\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 2 \\ 0 & 0 & 0 & 1 & 2 & -1 & 2 \end{array}\right] \begin{array}{l} R_1 \to R_1 - 2R_2 \\ \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 & 2 \end{array}\right] R_2 \to R_2 - \frac{1}{2}R_3$$

$$\left.\begin{aligned} x_1 + x_2 \qquad + x_5 - x_6 &= 0 \\ x_3 \quad - x_5 + x_6 &= 1 \\ x_4 + 2x_5 - x_6 &= 2 \end{aligned}\right\}$$

20

$$\left.\begin{aligned} x_1 &= -x_2 - x_5 + x_6 \\ x_2 &= x_2 \\ x_3 &= 1 + x_5 - x_6 \\ x_4 &= 2 - 2x_5 + x_6 \\ x_5 &= x_5 \\ x_6 &= x_6 \end{aligned}\right\}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -s \cancel{- t} + u \\ s \\ 1 + t - u \\ 2 - 2t + u \\ t \\ u \end{bmatrix} \quad \text{for } s, t, u \in \mathbb{R}$$

Problem 2. Determine whether there exists a polynomial of the form

$$f(t) = a + bt + ct^2 + dt^3$$

whose graph goes through the points $(0,1)$, $(1,0)$, $(-1,0)$, $(2,-15)$, $(-2,9)$ and $(3,-56)$. In the affirmative case, give $f(t)$.

$1 = a + 0 + 0 + 0 \Rightarrow a = 1$

$0 = a + b + c + d$

$0 = a - b + c - d$

$-15 = a + 2b + 4c + 8d$

$9 = a - 2b + 4c - 8d$

$-56 = a + 3b + 9c + 27d$

$$\begin{array}{cccc} b & c & d & \text{Solutions} \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 2 & 4 & 8 & -16 \\ -2 & 4 & -8 & 8 \\ 3 & 9 & 27 & -57 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 2 & 0 & -2 \\ 0 & 2 & 6 & -14 \\ 0 & 6 & -6 & 6 \\ 0 & 6 & 24 & -54 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 + 2R_1 \\ R_5 \to R_5 - 3R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 3 & -7 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 4 & -9 \end{array}\right] \begin{array}{l} \\ R_2 \to \tfrac{1}{2} R_2 \\ R_3 \to \tfrac{1}{2} R_3 \\ R_4 \to \tfrac{1}{6} R_4 \\ R_5 \to \tfrac{1}{6} R_5 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 4 & -8 \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_2 \\ \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \\ R_5 \to R_5 - R_2 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to \tfrac{1}{3} R_3 \\ R_4 \to -1 R_4 \\ R_5 \to \tfrac{1}{4} R_5 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_3 \\ \\ \\ R_4 \to R_4 - R_3 \\ R_5 \to R_5 - R_3 \end{array}$$

Yes there is a solution so the polynomial exists.

$$\begin{bmatrix} b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

and $a = 1$

$$\boxed{f(t) = 1 + 2t - t^2 - 2t^3}$$

30 **Problem 3.** Consider the matrix

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Recall that the linear transformation $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $R_\theta(\vec{x}) = A_\theta \vec{x}$ is a counter-clockwise rotation through an angle θ.

(a) Show that

$$R_\alpha \circ R_\beta = R_\beta \circ R_\alpha = R_{\alpha+\beta}$$

by calculation.

(b) Using the result of (a), show that R_α is invertible and describe R_α^{-1}.

Hint: For (a), you may use the trigonometric identities

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

a) $R_\alpha \circ R_\beta = A_\alpha A_\beta = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -(\cos\alpha\sin\beta - \sin\alpha\cos\beta) \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$ $*$

$R_\beta \circ R_\alpha = A_\beta A_\alpha = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha & -\cos\beta\sin\alpha - \sin\beta\cos\alpha \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha & -\sin\beta\sin\alpha + \cos\beta\cos\alpha \end{bmatrix}$ $**$

$R_{\alpha+\beta} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} \overset{\text{(By identities given)}}{=} \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -1(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix}$ $***$

$* = ** = ***$, so indeed $R_\alpha \circ R_\beta = R_\beta \circ R_\alpha = R_{(\alpha+\beta)}$ ✓

b) if R_α is invertible, then there exists R_α^{-1} such that $R_\alpha \circ R_\alpha^{-1} = \text{identity}$ and $R_\alpha^{-1} \circ R_\alpha = \text{identity}$.

We have $R_\alpha \circ R_\beta = R_{\alpha+\beta}$. If $\theta = 0$, $A_0 = \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{identity}$.

So if $\alpha+\beta = 0$, we have $R_\alpha \circ R_\beta = R_0 = \text{identity}$ and $R_\beta \circ R_\alpha = R_0 = \text{identity}$. So we found $R_\alpha^{-1} = R_\beta$ when $\alpha+\beta=0$. if $\alpha+\beta=0$, then $\beta = -\alpha$.

So $R_\alpha^{-1} = R_{(-\alpha)}$. (and R_α is invertible because of this.)

✓

Problem 4. Determine all the values (if any) of the constants B and C for which the following matrix is invertible:

$$\begin{bmatrix} 0 & 1 & B \\ -1 & 0 & C \\ -B & -C & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & B & 1 & 0 & 0 \\ -1 & 0 & C & 0 & 1 & 0 \\ -B & -C & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -C & 0 & -1 & 0 \\ -B & -C & 0 & 0 & 0 & 1 \\ 0 & 1 & B & 1 & 0 & 0 \end{array}\right] \begin{array}{l} -R_2 \\ R_3 \\ R_1 \end{array}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -C & 0 & -1 & 0 \\ 0 & -C & -BC & 0 & -B & 1 \\ 0 & 1 & B & 1 & 0 & 0 \end{array}\right] R_2 \to R_2 + BR_1 \to \left[\begin{array}{ccc|ccc} 1 & 0 & -C & 0 & -1 & 0 \\ 0 & 1 & B & 0 & B/C & 1/C \\ 0 & 1 & B & 1 & 0 & 0 \end{array}\right] R_2 \to \tfrac{-1}{C} R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -C & 0 & -1 & 0 \\ 0 & 1 & B & 0 & B/C & 1/C \\ 0 & 0 & 0 & 1 & -B/C & -1/C \end{array}\right] R_3 \to R_3 - R_2$$

There are no values of B and C which make this matrix invertible.