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Problem 1. Determine whether there exists a polynomial of the form

$$f(t) = a + bt + ct^2 + dt^3$$

whose graph goes through the points $(0,1)$, $(1,0)$, $(-1,0)$, $(2,-15)$, $(-2,9)$ and $(3,-56)$. In the affirmative case, give $f(t)$.

$a = 1$
$a+b+c+d = 0$
$a-b+c-d = 0$
$a+2b+4c+8d = -15$
$a-2b+4c-8d = 9$
$a+3b+9c+27d = -56$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 1 & 2 & 4 & 8 & -15 \\ 1 & -2 & 4 & -8 & 9 \\ 1 & 3 & 9 & 27 & -56 \end{bmatrix} \begin{array}{l} \\ -R_1 \\ -R_1 \\ -R_1 \\ -R_1 \\ -R_1 \end{array} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & -1 & -1 \\ 0 & 2 & 4 & 8 & -16 \\ 0 & -2 & 4 & 8 & 8 \\ 0 & 3 & 9 & 27 & -57 \end{bmatrix} \begin{array}{l} \\ \\ +R_2 \\ -2R_2 \\ +2R_2 \\ -3R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 2 & 6 & -14 \\ 0 & 0 & 6 & 10 & 6 \\ 0 & 0 & 6 & 24 & -54 \end{bmatrix} \quad \tfrac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 6 & -14 \\ 0 & 0 & 6 & 10 & 6 \\ 0 & 0 & 6 & 24 & -54 \end{bmatrix} \begin{array}{l} \\ \\ \\ -2R_3 \\ -6R_3 \\ -6R_3 \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 6 & -12 \\ 0 & 0 & 0 & 10 & 12 \\ 0 & 0 & 0 & 24 & -48 \end{bmatrix} \quad \tfrac{1}{6}\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 10 & 12 \\ 0 & 0 & 0 & 24 & -48 \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ -10R_4 \\ -24R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cancel{32} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$$ no solution, polynomial $\boxed{\text{~~does not~~ exist}}$

$0 \neq 32$

Problem 2. Consider the matrix

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Recall that the linear transformation $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $R_\theta(\vec{x}) = A_\theta \vec{x}$ is a counter-clockwise rotation through an angle θ.

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(a) Show that

$$R_\alpha \circ R_\beta = R_\beta \circ R_\alpha = R_{\alpha+\beta}$$

by calculation.

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(b) Using the result of (a), show that R_α is invertible and describe R_α^{-1}.

Hint: For (a), you may use the trigonometric identities

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

a)

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\begin{bmatrix} \cos\alpha\cos\beta + (-\sin\alpha)\sin\beta & \cos\alpha\cdot(-\sin\beta) + (-\sin\alpha)\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \sin\alpha(-\sin\beta) + \cos\alpha\cos\beta \end{bmatrix} \quad \begin{bmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha & -\cos\beta\sin\alpha - \sin\beta\cos\alpha \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha & -\sin\beta\sin\alpha + \cos\beta\cos\alpha \end{bmatrix}$$

$$\begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix} \checkmark$$

$$R_{\alpha+\beta} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$\boxed{\therefore R_\alpha \circ R_\beta = R_\beta \circ R_\alpha = R_{\alpha+\beta}}$$

b)

$$\left[\begin{array}{cc|cc} \cos\alpha & -\sin\alpha & 1 & 0 \\ \sin\alpha & \cos\alpha & 0 & 1 \end{array}\right] \begin{array}{l} /\cos\alpha \\ /\sin\alpha \end{array} \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{-\sin\alpha}{\cos\alpha} & 1/\cos\alpha & 0 \\ 1 & \frac{\cos\alpha}{\sin\alpha} & 0 & 1/\sin\alpha \end{array}\right] - R_2$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & \frac{-\sin\alpha}{\cos\alpha} & 1/\cos\alpha & 0 \\ 0 & \frac{\cos\alpha}{\sin\alpha} + \frac{\sin\alpha}{\cos\alpha} & 0 & 1/\sin\alpha \end{array}\right] (\cos\alpha\sin\alpha) \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{-\sin\alpha}{\cos\alpha} & 1/\cos\alpha & 0 \\ 0 & 2\cos\alpha\sin\alpha & 0 & \cos\alpha \end{array}\right] /2\cos\alpha\sin\alpha$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & \frac{-\sin\alpha}{\cos\alpha} & 1/\cos\alpha & 0 \\ 0 & 1 & 0 & \frac{1}{2\sin\alpha} \end{array}\right] + \frac{\sin\alpha}{\cos\alpha} R_2 \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1/\cos\alpha & \frac{1}{2\cos\alpha} \\ 0 & 1 & 0 & \frac{1}{2\sin\alpha} \end{array}\right]$$

$\boxed{R_\alpha^{-1} = \text{rotation by } \alpha \text{ clockwise}}$

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Problem 3. Determine all the values (if any) of the constants B and C for which the following matrix is invertible:

$$\begin{bmatrix} 0 & 1 & B \\ -1 & 0 & C \\ -B & -C & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & B & 1 & 0 & 0 \\ -1 & 0 & C & 0 & 1 & 0 \\ -B & -C & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\mathcal{Q}(-)} \left[\begin{array}{ccc|ccc} 1 & 0 & -C & 0 & -1 & 0 \\ 0 & 1 & B & 1 & 0 & 0 \\ -B & -C & 0 & 0 & 0 & 1 \end{array}\right] +BR_1$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -C & 0 & -1 & 0 \\ 0 & 1 & B & 1 & 0 & 0 \\ 0 & -C & -BC & 0 & -B & 1 \end{array}\right] +CR_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -C & 0 & -1 & 0 \\ 0 & 1 & B & 1 & 0 & 0 \\ 0 & 0 & 0 & C & -B & 1 \end{array}\right]$$

$\boxed{\text{not invertable}}$ the matrix doesn't fully reduce to the identity matrix.

$\boxed{\text{no values for } B \text{ \& } C}$

Problem 4. Let $A = \begin{bmatrix} 4 & 8 & 1 & 1 & 4 \\ 3 & 6 & 1 & 2 & 5 \\ 2 & 4 & 1 & 9 & 10 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix}$.

(a) Find a basis for $\ker(A)$.

(b) Find a basis for $\operatorname{im}(A)$.

$$\left[\begin{array}{ccccc|c} 4 & 8 & 1 & 1 & 4 & 0 \\ 3 & 6 & 1 & 2 & 5 & 0 \\ 2 & 4 & 1 & 9 & 10 & 0 \\ 1 & 2 & 3 & 2 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 2 & 0 & 0 \\ 3 & 6 & 1 & 2 & 5 & 0 \\ 2 & 4 & 1 & 9 & 10 & 0 \\ 4 & 8 & 1 & 1 & 4 & 0 \end{array}\right] \begin{array}{l} \\ -3R_1 \\ -2R_1 \\ -4R_1 \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 2 & 0 & 0 \\ 0 & 0 & -8 & -4 & 5 & 0 \\ 0 & 0 & -5 & 5 & 0 & 0 \\ 0 & 0 & -11 & -7 & 4 & 0 \end{array}\right] /-5$$

$$\to \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -8 & -4 & 5 & 0 \\ 0 & 0 & -11 & -7 & 4 & 0 \end{array}\right] \begin{array}{l} -3R_2 \\ \\ +8R_2 \\ +11R_2 \end{array} \to \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -12 & 5 & 0 \\ 0 & 0 & 0 & -18 & 4 & 0 \end{array}\right] /-18 \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2/9 & 0 \\ 0 & 0 & 0 & -12 & 5 & 0 \end{array}\right] \begin{array}{l} -5R_3 \\ +R_3 \\ \\ +12R_3 \end{array}$$

$$\to \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 10/9 & 0 \\ 0 & 0 & 1 & 0 & -2/9 & 0 \\ 0 & 0 & 0 & 1 & -2/9 & 0 \\ 0 & 0 & 0 & 0 & -24/9 & 0 \end{array}\right] /-24/9 \to \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 10/9 & 0 \\ 0 & 0 & 1 & 0 & -2/9 & 0 \\ 0 & 0 & 0 & 1 & -2/9 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} -10/9\,R_4 \\ +2/9\,R_4 \\ +2/9\,R_4 \\ \\ \end{array}$$

$$\to \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \to \begin{array}{l} x_1 + 2x_2 = 0 \Rightarrow x = -2x_2 \\ x_3 = 0 \\ x_4 = 0 \\ x_5 = 0 \end{array} \Rightarrow \begin{bmatrix} -2t \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Basis $\ker(A) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Basis $\operatorname{im}(A)$ = columns in A w/ leading 1's in $\operatorname{rref}(A)$ (1, 3, 4, 5)

Basis $\operatorname{im}(A) = \left\{ \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 10 \\ 0 \end{bmatrix} \right\}$

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