(1) $f(t) = a + bt + ct^2 + dt^3$ whose graph goes through $(0,1), (1,0), (-1,0), (2,-15),$ $(-2,9)$ and $(3,-56)$.

$$\left.\begin{array}{lll}
(0,1) \Rightarrow f(0)=1 & \Rightarrow & a = 1 \\
(1,0) \Rightarrow f(1)=0 & \Rightarrow & a+b+c+d=0 \\
(-1,0) \Rightarrow f(-1)=0 & \Rightarrow & a-b+c-d=0 \\
(2,-15) \Rightarrow f(2)=-15 & \Rightarrow & a+2b+4c+8d=-15 \\
(-2,9) \Rightarrow f(-2)=9 & \Rightarrow & a-2b+4c-8d=9 \\
(3,-56) \Rightarrow f(3)=-56 & \Rightarrow & a+3b+9c+27d=-56
\end{array}\right\}$$

$$\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
1 & -1 & 1 & -1 & 0 \\
1 & 2 & 4 & 8 & -15 \\
1 & -2 & 4 & -8 & 9 \\
1 & 3 & 9 & 27 & -56
\end{array}\right]
\begin{array}{l} \\ R_2-R_1 \\ R_3-R_1 \\ R_4-R_1 \\ R_5-R_1 \\ R_6-R_1 \end{array}
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & -1 \\
0 & -1 & 1 & -1 & -1 \\
0 & 2 & 4 & 8 & -16 \\
0 & -2 & 4 & -8 & 8 \\
0 & 3 & 9 & 27 & -57
\end{array}\right]
\begin{array}{l} \\ \\ R_3+R_2 \\ R_4-2R_2 \\ R_5+2R_2 \\ R_6-3R_2 \end{array}
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & -1 \\
0 & 0 & 2 & 0 & -2 \\
0 & 0 & 2 & 6 & -14 \\
0 & 0 & 6 & -6 & 6 \\
0 & 0 & 6 & 24 & -54
\end{array}\right]$$

$$\begin{array}{r} \\ \\ \frac{1}{2}R_3 \\ \frac{1}{2}R_4 \\ \frac{1}{6}R_5 \\ \frac{1}{6}R_6 \end{array}
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & -1 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 3 & -7 \\
0 & 0 & 1 & -1 & 1 \\
0 & 0 & 1 & 4 & -9
\end{array}\right]
\begin{array}{l} \\ R_2-R_3 \\ \\ R_4-R_3 \\ R_5-R_3 \\ R_6-R_3 \end{array}
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 3 & -6 \\
0 & 0 & 0 & -1 & 2 \\
0 & 0 & 0 & 4 & -8
\end{array}\right]$$

$$\begin{array}{r} \\ \\ \\ \frac{1}{3}R_4 \\ \\ \\ \end{array}
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & -1 & 2 \\
0 & 0 & 0 & 4 & -8
\end{array}\right]
\begin{array}{l} \\ R_2-R_5 \\ \\ \\ R_5+R_4 \\ R_6-4R_4 \end{array}
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
\Rightarrow a + bt + ct^2 + dt^3$$

$$\parallel$$

$$f(t) = \boxed{1 + 2t - t^2 - 2t^3}$$

(2)

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(a) ~~$R_\alpha R_\beta$~~ It suffices to show $R_\alpha \circ R_\beta = R_{\alpha+\beta}$

$$\Longleftrightarrow \quad A_\alpha \cdot A_\beta = A_{\alpha+\beta}.$$

$$A_\alpha \cdot A_\beta = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha \cdot \cos\beta - \sin\alpha \sin\beta & -\cos\alpha \sin\beta - \sin\alpha \cos\beta \\ \sin\alpha \cos\beta + \cos\alpha \cdot \sin\beta & -\sin\alpha \sin\beta + \cos\alpha \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$= A_{\alpha+\beta}.$$

(b) ~~taking~~ Since $R_\theta = id$ for $\theta = 0$,

taking $\beta = -\alpha$ in (a) we have:

$$R_\alpha \circ R_{(-\alpha)} = R_0 = id$$

$$\Rightarrow R_\alpha^{-1} = R_{(-\alpha)}$$: rotation through an angle θ in clockwise direction.

(3)

$$\left[\begin{array}{ccc|ccc} 0 & 1 & B & 1 & 0 & 0 \\ -1 & 0 & C & 0 & 1 & 0 \\ -B & -C & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} +1 & 0 & -C & 0 & -1 & 0 \\ 0 & 1 & B & 1 & 0 & 0 \\ -B & -C & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 + BR_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -C & 0 & -1 & 0 \\ 0 & 1 & B & 1 & 0 & 0 \\ 0 & -C & -BC & 0 & -B & 1 \end{array}\right] \xrightarrow{R_3 + CR_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -C & 0 & -1 & 0 \\ 0 & 1 & B & 1 & 0 & 0 \\ 0 & 0 & 0 & C & -B & 1 \end{array}\right]$$

$\therefore$ NOT invertible for any values of B, C.

(4)

$$A = \begin{bmatrix} 4 & 8 & 1 & 1 & 4 \\ 3 & 6 & 1 & 2 & 5 \\ 2 & 4 & 1 & 9 & 10 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix}$$

(a)

$$\begin{array}{c} \\ R_1 \leftrightarrow R_4 \\ \\ \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 2 & 0 & 0 \\ 3 & 6 & 1 & 2 & 5 & 0 \\ 2 & 4 & 1 & 9 & 10 & 0 \\ 4 & 8 & 1 & 1 & 4 & 0 \end{array}\right] \begin{array}{c} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \\ R_4 - 4R_1 \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 2 & 0 & 0 \\ 0 & 0 & -8 & -4 & 5 & 0 \\ 0 & 0 & -5 & 5 & 10 & 0 \\ 0 & 0 & -11 & -7 & 4 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ R_2 \leftrightarrow \frac{1}{-5}R_3 \\ \\ \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & -8 & -4 & 5 & 0 \\ 0 & 0 & -11 & -7 & 4 & 0 \end{array}\right] \begin{array}{c} R_1 - 3R_2 \\ \\ R_3 + 8R_2 \\ R_4 + 11R_2 \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 5 & +6 & 0 \\ 0 & 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & -12 & -11 & 0 \\ 0 & 0 & 0 & -18 & -18 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \\ \\ R_3 \leftrightarrow \frac{1}{-18}R_4 \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 5 & 6 & 0 \\ 0 & 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -12 & -11 & 0 \end{array}\right] \begin{array}{c} R_1 - 5R_3 \\ R_2 + R_3 \\ \\ R_4 + 12R_3 \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_4 \\ R_2 + R_4 \\ R_3 - R_4 \\ \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \qquad \left.\begin{array}{r} x_1 + 2x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \\ x_5 = 0 \end{array}\right\} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} t$$

$\therefore$ basis for $\ker(A)$: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

(b) Basis for $\operatorname{im}(A)$: $\left\{ \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 10 \\ 0 \end{bmatrix} \right\}$.