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Math 33A, Lecture 2
Winter 2016
02/17/16
Time Limit: 50 Minutes

Name (Print): [redacted]
SID Number:

Day \ T.A.	Bon-Soon	David	Robert
Tuesday	2A	2C	2E
Thursday	2B	2D	2F

This exam contains 10 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, cross the box corresponding to your discussion section, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Use a pen to record your **final answers**. Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the last two pages for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the last two pages; clearly indicate when you have done this.

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Problem	Points	Score	
1	8	7	
2	10	8	RH
3	10	10	RH
4	11	8	
5	11	7	
Total:	50	~~37~~ 40	RH.

1. (8 points) For each of the following statements, circle T for True, F for False.

T ~~F~~ If A, B are two $n \times n$ symmetric matrices then their product AB is also symmetric.

T F For every $n \times m$ matrix A, $ker(A)^{\perp} = im(A^T)$.

T F For every two $n \times n$ matrices A, B, $rank(AB) \leq rank(B)$.

T F If Q is an orthogonal matrix then so is $-Q^T$.

T F If $V \subset \mathbb{R}^6$ is a subspace and $dim(V) = 3$ then the matrix representing the projection onto V is a 3×3 matrix.

T F For each $n \geq 1$, there exists an $n \times n$ matrix A which is both symmetric and skew-symmetric.

invertible

T F There is a 3×3 matrix A so that $det(A) \neq 0$ and $im(A) \neq \mathbb{R}^3$.

T F If x, y are two positive real numbers and $x + y \leq 2$ then $\frac{1}{x} + \frac{1}{y} \geq 2$.

2. (10 points) Let $\mathfrak{B}$ be a basis for $\mathbb{R}^2$ given by the vectors $v_1 = \begin{bmatrix}1\\2\end{bmatrix}, v_2 = \begin{bmatrix}4\\3\end{bmatrix}$

(a) Find the coordinates of the vectors $w_1 = \begin{bmatrix}3\\1\end{bmatrix}$ and $w_2 = \begin{bmatrix}3\\-4\end{bmatrix}$ in the basis $\mathfrak{B}$.

(b) Let $T(\vec{x}) = proj_L(\vec{x})$ be the projection to the line L passing though the origin and $\begin{bmatrix}3\\-4\end{bmatrix}$.

Find the $\mathfrak{B}$−matrix of T.

(a) $\vec{w}_1 = \begin{pmatrix}3\\1\end{pmatrix} = -\vec{v}_1 + \vec{v}_2$. Therefore $\vec{w}_1$ is $\begin{pmatrix}-1\\1\end{pmatrix}$ in $\mathbb{B}$. $\begin{pmatrix}-1\\1\end{pmatrix}$ 5

$\vec{w}_2 = \begin{pmatrix}3\\-4\end{pmatrix} = -5\vec{v}_1 + 2\vec{v}_2$. Therefore $\vec{w}_2$ is $\begin{pmatrix}-5\\2\end{pmatrix}$ in $\mathbb{B}$. $\begin{pmatrix}-5\\2\end{pmatrix}$

(b) Let $\vec{u} = \frac{1}{5}\begin{pmatrix}3\\-4\end{pmatrix}$.

Then
$$T(\vec{x}) = (\vec{x}\cdot\vec{u})\vec{u}$$
$$= (x_1u_1 + x_2u_2)\vec{u}$$
$$= \begin{pmatrix}x_1u_1^2 + x_2u_1u_2\\ x_1u_1u_2 + x_2u_2^2\end{pmatrix}$$
$$= \begin{bmatrix}u_1^2 & u_1u_2\\ u_1u_2 & u_2^2\end{bmatrix}\vec{x}$$
$$= \frac{1}{25}\begin{bmatrix}9 & -12\\ -12 & 16\end{bmatrix}\vec{x}$$

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Then the $\mathbb{B}$-matrix of T is
$$\begin{bmatrix} | & | \\ [T(\vec{v}_1)]_{\mathbb{B}} & [T(\vec{v}_2)]_{\mathbb{B}} \\ | & | \end{bmatrix} = \begin{bmatrix} -1 & 0\ \cancel{\frac{48}{125}} \\ \frac{2}{5} & 0\ \cancel{-\frac{12}{125}} \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0\ \cancel{\frac{48}{125}} \\ \frac{2}{5} & 0\ \cancel{-\frac{12}{125}} \end{bmatrix}$$

$$* \quad T(\vec{v}_1) = \frac{1}{25}\left(\begin{pmatrix}9\\-12\end{pmatrix} + 2\begin{pmatrix}-12\\16\end{pmatrix}\right)$$
$$= \frac{1}{25}\begin{pmatrix}-15\\20\end{pmatrix}$$
$$= \frac{1}{5}\begin{pmatrix}-3\\4\end{pmatrix} = \frac{1}{5}(-5\vec{v}_1 + 2\vec{v}_2) = -\vec{v}_1 + \frac{2}{5}\vec{v}_2$$

$$* \quad T(\vec{v}_2) = \frac{1}{25}\left(4\begin{pmatrix}9\\-12\end{pmatrix} + 3\begin{pmatrix}-12\\16\end{pmatrix}\right)$$
$$= \frac{1}{25}\begin{pmatrix}0\\ \cancel{12}\ 0\end{pmatrix} \cancel{= \frac{12}{25}\cdot\frac{1}{5}(4\vec{v}_1 - \vec{v}_2) = \frac{48}{125}\vec{v}_1 - \frac{12}{125}\vec{v}_2}$$
$$= 0\vec{v}_1 + 0\vec{v}_2$$

3. (10 points)

6/6

(a) Let $A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $V = im(A) \subset \mathbb{R}^4$. Find a basis for the orthogonal complement $V^\perp$. What is $\dim(V)$?

4/4

(b) Compute the following determinants:

$$det \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}, det \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 4 & 0 & 0 \end{bmatrix}.$$

(a) $V^\perp = im(A)^\perp = ker(A^T)$

$$= ker \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ -1 & 2 & 1 & 1 & 0 \\ 3 & -2 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} \\ +R_1 \\ -3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & -2 & -2 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \\ +R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \cdot \frac{1}{2} \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{cases} x_1 + x_3 + x_4 = 0 \\ x_2 + x_3 + x_4 = 0 \end{cases}$$

$$\begin{cases} x_1 = -x_3 - x_4 \\ x_2 = -x_3 - x_4 \\ x_3 = x_3 \\ x_4 = \quad x_4 \end{cases} \Rightarrow \text{a basis for } V^\perp \text{ is } \left\langle \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle \qquad \left\langle \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

$\dim(V) = n - \dim(V^\perp) = 4 - 2 = 2$ $\qquad \dim(V) = 2$

Yes.

(b) (i)

$$det \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{matrix} 1 & -1 \\ 1 & 1 \\ 2 & 1 \end{matrix}$$

$= -(0) - (2) - (-3) + (3) + (-4) + (0)$

$= -2 + 3 + 3 - 4$

$= 1 - 1$

$= 0$

0

(ii)

$$det \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 4 & 0 & 0 \end{bmatrix} \begin{matrix} 0 & 0 \\ 0 & 1 \\ 4 & 0 \end{matrix}$$

$= -(-8) - (0) - (0) + (0) + (0) + (0)$

$= 8$

8

4. (11 points)

6

(a) Find the least-squares solution $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ -1 & 1 \\ 0 & 5 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

e 0

(b) Let $A\vec{x} = \vec{b}$ be a linear inconsistent system and suppose that $\vec{b} \in im(A)^{\perp}$. Explain why $\vec{x}^* = \vec{0}$ is a least squares solution of this system.

(a) $A^T A \vec{x}^* = A^T \vec{b}$

$\begin{bmatrix} 6 & 0 \\ 0 & 28 \end{bmatrix} \vec{x} = \left\langle \begin{matrix} 2 \\ 6 \end{matrix} \right\rangle$ (see scratch)

$\left[\begin{matrix} 6 & 0 \\ 0 & 28 \end{matrix}\right] \begin{matrix} 2 \\ 6 \end{matrix} \quad \begin{matrix} \cdot\frac{1}{6} \\ \cdot\frac{1}{28} \end{matrix}$

$\left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right] \begin{matrix} \frac{1}{3} \\ \frac{3}{14} \end{matrix} \qquad \vec{x}^* = \left\langle \begin{matrix} \frac{1}{3} \\ \frac{3}{14} \end{matrix} \right\rangle \qquad \vec{x}^* = \left\langle \frac{1}{3}, \frac{3}{14} \right\rangle$

$\vec{x}^*$ doesn't need to be in $im(A)$. $A\vec{x}^*$ does.

(b) If $\vec{b} \in im(A)^{\perp}$, then there isn't really an $A\vec{x}^* \in im(A)$ that can approximate $\vec{b}$. The best we can do is minimize the error $\|\vec{b} - A\vec{x}^*\|$; this leads us to $\vec{x}^* = \vec{0}$, which is the projection of $\vec{b}$ onto $im(A)$.

incorrect argument

not in prec. -5

e

2/5: Robert [illegible]

5. (11 points)

7 (a) Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$. Use the Gram-Schmidt process to compute an orthonormal basis for $im(A)$.

0 (b) Can there exist two 3×3 matrices A, B so that $det(A) = det(B)$ and $A - B$ is invertible? Justify your answer by either giving an example of such matrices or explain why they cannot exist.

(a) $\vec{v}_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_2 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \vec{v}_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$

$* \|\vec{v}_1\| = \sqrt{4} = 2$

$\vec{u}_1\ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_2^\perp \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \vec{v}_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$

$* \vec{v}_2^\perp = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$

$= \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{4}\left(\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \cancel{- \frac{1}{4}(0)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}$

$= \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

$* \|\vec{v}_2^\perp\| = \sqrt{4} = 2$

$\vec{u}_1\ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \vec{u}_2\ \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \vec{v}_3^\perp \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

$* \vec{v}_3^\perp = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$

$= \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} - \frac{1}{4}\left(\begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{4}\left(\begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}\right)\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} - \frac{1}{4}(2)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{4}(-2)\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

$* \|\vec{v}_3^\perp\| = \sqrt{4} = 2$

$\vec{u}_1\ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \vec{u}_2\ \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \vec{u}_3\ \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

Im(A): orthonormal basis

$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

no part (b).

Extra paper 1

4a) A^TA :

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ -1 & 1 \\ 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 28 \end{bmatrix}$$

$A^T\vec{b}$: $1\left\langle \begin{matrix} \cancel{1} \\ \cancel{-1} \end{matrix} \right\rangle + 1\left\langle \begin{matrix} 2 \\ \cancel{1} \end{matrix} \right\rangle + 1\left\langle \begin{matrix} \cancel{-1} \\ 1 \end{matrix} \right\rangle + 1\left\langle \begin{matrix} 0 \\ 5 \end{matrix} \right\rangle$

$= \left\langle \begin{matrix} 2 \\ 6 \end{matrix} \right\rangle$

Extra paper 2