Math 33A, Lecture 2
Winter 2016
01/25/16
Time Limit: 50 Minutes

Name (Print):
SID Number:

Day \ T.A.	Bon-Soon	David	Robert
Tuesday	2A	2C	2E
Thursday	2B	2D	2F

This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, cross the box corresponding to your discussion section, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Use a pen to record your **final answers**. Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the last page and the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Problem	Points	Score
1	8	6
2	10	10
3	10	10
4	10	10
5	12	6
Total:	50	42

1. (8 points) For each of the following statements, circle T for True, F for False.

T F If L_1 and L_2 are two perpendicular lines in $\mathbb{R}^2$ then for every $\vec{x} \in \mathbb{R}^2$,

$$Proj_{L_1}(\vec{x}) + Proj_{L_2}(\vec{x}) = \vec{x}$$

T F For every $n \times n$ matrix A, if A^3 is invertible then so is A.

$A^3B = I_n$

$\Downarrow$

$\underbrace{A(A^2B)}_{A^{-1}} = I_n$

T F If W_1, W_2 are two subspaces of $\mathbb{R}^n$ then their intersection $W_1 \cap W_2$ is also a subspace of $\mathbb{R}^n$.

T F The function $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 - x_3 \\ x_1x_3 \\ x_1 - x_2 \end{bmatrix}$$

not linear

is a linear transformation.

T F The matrix $\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$ is in reduced row echelon form.

T F For every $n \times n$ matrix A, if $ker(A) = \{\vec{0}\}$ then the columns of A are linearly independent.

T F There are two distinct 2×2 matrices $A \neq B$ such that $AB = A + B$.

T F The matrix $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ represents a reflection.

reflection combined with scaling

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$\begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} = \begin{bmatrix} a + e & b + f \\ c + g & d + h \end{bmatrix}$$

$$\begin{cases} ae + bg = a + e \\ af + bh = b + f \\ ce + dg = c + g \\ cf + dh = d + h \end{cases} \quad \text{let } a, d, e, h = 1$$

$$\begin{cases} 1 + bg = 1 + 1 \\ \cancel{f + b = b + f} \\ \cancel{c + g = c + g} \\ cf + 1 = 1 + 1 \end{cases}$$

$$\begin{cases} bg = 1 \\ cf = 1 \end{cases} \quad \text{let } b, c = \tfrac{1}{2} \text{ and } g, f = 2$$

Then $A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. $AB = A + B \overset{\checkmark}{=} \begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 2 \end{bmatrix}$.

2. (10 points)

(a) Compute the inverse of $\begin{bmatrix} 1 & 5 & 3 \\ 2 & 11 & 6 \\ 1 & 5 & 5 \end{bmatrix}$.

(b) Let A be a 3×3 matrix and $\vec{b}$ be a vector in $\mathbb{R}^3$. Suppose that the set of solutions $\{\vec{x} \in \mathbb{R}^3 \mid A\vec{x} = \vec{b}\}$ is a line in $\mathbb{R}^3$. What is the rank of A?

(a)
$$\left[\begin{array}{ccc|ccc} 1 & 5 & 3 & 1 & 0 & 0 \\ 2 & 11 & 6 & 0 & 1 & 0 \\ 1 & 5 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -2R_1 \\ -R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} -5R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 11 & -5 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \cdot\frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 11 & -5 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right] \begin{array}{l} -3R_3 \\ \\ \\ \end{array}$$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \begin{bmatrix} \frac{25}{2} & -5 & -\frac{3}{2} \\ -2 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

(a) $\begin{bmatrix} \frac{25}{2} & -5 & -\frac{3}{2} \\ -2 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

(b) If the set of solutions is a line, then A has one free variable.
In which case rank(A) = (# colums) − (# free vars) = 3 − 1 = 2

(b) 2

3. (10 points) Let $A = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ 1 & -2 & 2 & 3 & 5 \\ 2 & -4 & 1 & 0 & 2 \end{bmatrix}$.

(a) Find a basis for the image of A.

(b) Find a basis for the kernel of A.

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 1 & 2 & 0 \\ 1 & -2 & 2 & 3 & 5 & 0 \\ 2 & -4 & 1 & 0 & 2 & 0 \end{array}\right] \begin{array}{l} \\ -R_1 \\ -2R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & -1 & -2 & -2 & 0 \end{array}\right] \begin{array}{l} -R_2 \\ \\ +R_2 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} +R_3 \\ -3R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{cases} x_1 - 2x_2 - x_4 = 0 \\ x_3 + 2x_4 = 0 \\ x_5 = 0 \end{cases}$$

$$\begin{cases} x_1 = 2x_2 + x_4 \\ x_2 = x_2 \\ x_3 = -2x_4 \\ x_4 = x_4 \\ x_5 = 0 \end{cases}$$

P P

(x_2, x_4 are arb. real numbers)

(a) basis of im (A) =

~~Span~~ $\left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} \right)$ ✓

(b) basis of ker (A) =

~~Span~~ $\left(\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right)$ ✓

(a) basis of im (A) =

~~Span~~ $\left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} \right)$ ✓

(b) basis of ker (A) =

~~Span~~ $\left(\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right)$ ✓

/10

4. (10 points) You may use geometric reasoning to justify your answers to the following questions.

6/6 (a) Find $ker(A)$ and $im(A)$ where A is the matrix representing the projection to the line $x_1 - x_2 = 0$ in $\mathbb{R}^2$.

4/4 (b) Let B be the matrix representing the reflection at about the line $x_1 + 2x_2 = 0$ in $\mathbb{R}^2$. Find B^2.

(a) "$x_1 - x_2 = 0$" refers to the line $y = x$.

$A\vec{x} = \vec{0}$ iff $\vec{x}$ has no component parallel to $y = x$.
Therefore the kernel consists of vectors perpendicular to $y = x$:

$\text{Ker}(A) = \text{span}\left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle$ (a) $\ker(A) = \text{Span}\left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle$

Projecting any vector onto $y = x$ will result in a vector on that line. Therefore the image consists of vectors on $y = x$:

$\text{im}(A) = \text{span}\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$ $\text{im}(A) = \text{span}\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$

(b) $B^2 = I_2$. Repeating any reflection results in the original vector.

(b) $B^2 = I_2$

5. (12 points)

6 (a) Give an example of a linear transformation whose kernel is the plane $3x + 3y + 2z = 0$ in $\mathbb{R}^3$.

0 (b) Let A, B be two $n \times n$ invertible matrices. Suppose that the system $(A - B)\vec{x} = \vec{0}$ has a solution $\vec{u} \neq \vec{0}$. Show that the matrix $C = I_n - B^{-1}A$ is not invertible.

(a) $[3\ 3\ 2]$

(a) $[3\ 3\ 2]$

why ?

6/7

The kernel of

$T(\vec{x}) = [3\ 3\ 2]\vec{x} = 0$

consists of $\vec{x}$ s.t.

$$[3\ 3\ 2]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$3x + 3y + 2z = 0$

So this plane makes up the kernel of T.

part (b) Missing

0/5

(b) $(A-B)\vec{x} = \vec{0}$ has a solution $\vec{x} \neq 0$

$\Rightarrow \ker(A-B)$ contains an $\vec{x} \neq 0$

$C = I_n - B^{-1}A$

$C = B^{-1}B - B^{-1}A$

$C = -B^{-1}(A-B)$

$\ker(C) = \ker(B^{-1}(A-B))$

$\ker(C) \supseteq \ker(A-B)$

$\ker(C)$ contains an $\vec{x} \neq 0$

$\Rightarrow \operatorname{rank}(C) < n$

$\Rightarrow \operatorname{rref}(C) \neq I_n$

$\Rightarrow C$ not invertible

Extra scratch paper