Math 33A - Midterm 2

Name: Sara Choi

UID: 704975729

Discussion session: 1B

Problems	Points	Score
1	35	35
2	30	24
3	25	12
4	10	5
Total	100	76

Problem 1. *(35 points)* Consider the linear transformation

35

$$f(x, y) = (x + 2y, -x + 3y).$$

(a) *(15 points)* Find a basis for the image of f, that is for $Im(f)$.
Hint: Find a 2×2 matrix A such that $f(X) = AX$ for all $X \in \mathbb{R}^2$, and find a basis for $\mathcal{R}(A)$, the range of A.

(b) *(20 points)* For the basis $\mathcal{B}$ of $\mathbb{R}^2$ given by

$$\mathcal{B} = \left\{ v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\},$$

find a 2×2 matrix $A_{\mathcal{B}}$ such that

$$[f(X)]_{\mathcal{B}} = A_{\mathcal{B}}[X]_{\mathcal{B}} \text{ for all } X \in \mathbb{R}^2.$$

a)

$$A\begin{bmatrix} \cancel{x} \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ -x+3y \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \quad rref(A) = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{array}{r} 1\ 2 \\ -\ 0\ 2 \\ \hline 1\ 0 \end{array}$$

basis for ✓

$$\boxed{im(f) = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}}$$

would also prefer people not use "="

b)

$$S = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \quad S^{-1} = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 2 & -1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right]$$

$$B = S^{-1}AS$$

$$= \begin{bmatrix} 2 & -1 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ -1 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 4 & 10 \\ -1/2 & 0 \end{bmatrix}}$$ ✓

Problem 2. *(30 points)*

(a) *(20 points)* Construct an orthonormal basis of $\mathbb{R}^2$ from the vectors

$$v_1 = \begin{bmatrix}1\\1\end{bmatrix}, \quad v_2 = \begin{bmatrix}2\\4\end{bmatrix}.$$

18

(b) *(10 points)* Is the set

$$V = \{(x,y) \in \mathbb{R}^2 \mid y^2 = x^4\}$$

6

a subspace of $\mathbb{R}^2$? Justify your answer.

a)

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{1+1}}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$

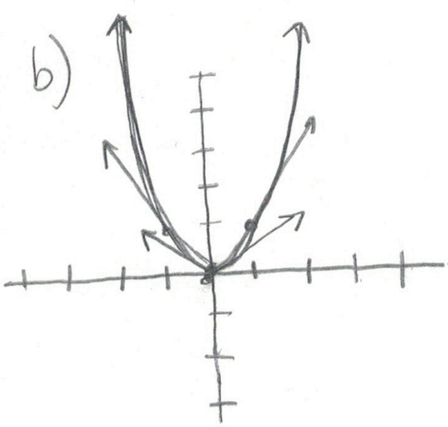
$$\vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp} = \frac{1}{\sqrt{1+9}}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}1/\sqrt{10}\\3/\sqrt{10}\end{bmatrix}$$

$$-\vec{u}_2(\vec{u}_2\cdot\vec{v}_2)$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{u}_1(\vec{u}_1\cdot\vec{v}_2) = \begin{bmatrix}2\\4\end{bmatrix} - \left((\sqrt{2})\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}\right) = \begin{bmatrix}2\\4\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix} =$$

$$\boxed{\left\{\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}, \begin{bmatrix}1/\sqrt{10}\\3/\sqrt{10}\end{bmatrix}\right\}}$$

b)



addition

No. Not closed under ~~scalar multiplication~~.

no negative numbers only in 1st quadrant

$$t_1^2 = t_2^4$$

$$t_1^2 - t_2^4 = 0$$

$$X = (x_1, x_2)$$

$$y = (y_1, y_2)$$

$$(\lambda^2 x_1^2 + 2\lambda x_1 \mu y_1 + \mu^2 y_1^2) - (\lambda^2 x_2^2 + 2\lambda x_2 \mu y_2 + \mu^2$$

$$\neq 0$$

$$\lambda\vec{x} + \mu\vec{y} \in \mathbb{R}^2$$

$$\begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$$

$\in V \quad \in V \quad \notin V$

$$(\lambda x_1, \lambda x_2) + (\mu y_1, \mu y_2)$$

$$(\underbrace{\lambda x_1 + \mu y_1}_{t_1}, \underbrace{\lambda x_2 + \mu y_2}_{t_2})$$

Problem 3. *(25 points)*

(a) *(15 points)* Assume that A is an invertible $n \times n$ matrix. Describe the kernel (nullspace) of its inverse A^{-1}:

$$\mathcal{N}(A^{-1}).$$

(b) *(10 points)* Does there exist a 3×3 matrix A such that

$$dim(\mathcal{R}(A)) = 2 \text{ and } dim(\mathcal{N}(A)) = 2?$$

Justify your answer.

Hint: Use the rank-nullity Theorem.

a) kernel must be $\{\vec{0}\}$

$$\text{rref}(A) = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{\text{all linearly independent}}$$

$c_1\vec{v}_1 + c_2\vec{v}_2 \dots c_m\vec{v}_m = 0$

$\hookrightarrow c_1 = c_2 = \dots = c_m = 0$

But this doesn't show kernel is 0:

$A^{-1}x = 0 \Rightarrow x = 0$

after mult. by A.

$\mathcal{N}(A^{-1}) = \{\vec{x} : A^{-1}\vec{x} = \vec{0}\}$

$A(A^{-1}\vec{x}) = A\vec{0}$

$\mathcal{N}(A^{-1}) = \{\vec{0}\}$

$\text{rank}(A^{-1}) = n$

$\dim(\ker(A)) + \dim(\text{im}(A)) = n$

b) ✓ $\dim(\mathcal{R}(A)) + \dim(\mathcal{N}(A)) = 3$

$2 + 2 \neq 3$

Such a matrix does not exist.

5

Problem 4. *(10 points)* Let A be an $n \times n$ matrix, and let S be an **invertible** $n \times n$ matrix. Show that

$$\mathcal{N}(A) = \mathcal{N}(SA).$$

Hint: Show separately that $\mathcal{N}(A) \subseteq \mathcal{N}(SA)$ (by considering some $x \in \mathcal{N}(A)$ and showing that we have also that $x \in \mathcal{N}(SA)$) and that $\mathcal{N}(SA) \subseteq \mathcal{N}(A)$ (by arguing similarly as in the previous case).

$A\vec{x} = \vec{0}$

$SA\vec{x} \stackrel{?}{=} \vec{0}$

$S(A\vec{x}) \stackrel{?}{=} \vec{0}$

$S(\vec{0}) = \vec{0}$

$A\vec{x} = SA\vec{x} = 0$

$\vec{x}$ exists in $\mathcal{N}(A)$ and $\mathcal{N}(SA)$. $\Rightarrow \mathcal{N}(A) \subseteq \mathcal{N}(SA)$

~~Since $\mathcal{N}(SA) \subseteq \mathcal{N}(A)$, $\mathcal{N}(A) = \mathcal{N}(SA)$~~ ? -5

$\vec{x} \in N(SA)$

$SA\vec{x} = \vec{0}$

$S^{-1}(SA\vec{x}) = S^{-1}\vec{0}$

$A\vec{x} = \vec{0} \Rightarrow \mathcal{N}(SA) \subseteq \mathcal{N}(A)$

$\therefore \mathcal{N}(A) = \mathcal{N}(SA)$