MATH 33A - MIDTERM 1

Name: Key

UID:

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Problem 1. *(25 points)*

(a) *(10 points)* Let A be a square and invertible matrix. Show that
$$(A^T)^{-1} = (A^{-1})^T.$$

(b) *(15 points)* Find the inverse of the matrix
$$B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}.$$

a.) We need to check that $A^T(A^{-1})^T = (A^{-1})^T A^T = I$.
We have $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$. Similarly, we see $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$.

b.) Look at $[B \mid I]$ and see if left side row reduces to I.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right].$$

So $$\boxed{B^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}}.$$

Problem 2. *(25 points)*

(a) *(10 points)* Find the matrix of the linear transformation that first projects a vector $v \in \mathbb{R}^2$ on the line $y = x$, then rotates it counterclockwise by a $\pi/4$ angle, and then scales it by a factor $k > 0$.

(b) *(15 points)* Let w be a vector on a line L in $\mathbb{R}^2$ that passes through the origin. Consider the $proj_L(v)$ which is the projection of another vector $v \in \mathbb{R}^2$ onto L. Finally let A be the 2×2 matrix whose columns are the vectors w and $proj_L(v)$. Is A invertible? Justify your answer.

a.) Let A_1 = projection, A_2 = rotation, A_3 = Scaling.

We are looking for $A_3 A_2 A_1$.

A_1: a unit vector along L is $\vec{u} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. So

$A_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = proj_L \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left(\vec{u} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \vec{u} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$. Similarly

$A_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ So $A_1 = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

A_2: Plug into rotation matrix formula: $\begin{bmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{bmatrix}$

$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

A_3: Just $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$.

$$A_3 A_2 A_1 = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 \\ k/\sqrt{2} & k/\sqrt{2} \end{bmatrix}}$$

b.) Since $\vec{w}$ and $proj_L(\vec{v})$ are on same line, they differ by a scalar multiple. So rref(A) has columns that differ by a scalar multiple, so $rref(A) \neq I$ and therefore A is <u>not</u> invertible.

Problem 3. *(25 points)*

(a) *(10 points)* Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with the property that

$$T\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = \begin{bmatrix}9\\3\end{bmatrix} \quad T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}2\\4\end{bmatrix}.$$

(b) *(15 points)* Let A be a given 3×3 matrix, and v be a given vector in $\mathbb{R}^3$. Is the following transformation $F : \mathbb{R}^3 \to \mathbb{R}^3$ that is given by

$$F(y) = v \times y + Ay \text{ for all } y \in \mathbb{R}^3,$$

where

$$v \times y = \begin{bmatrix} v_2y_3 - v_3y_2 \\ v_3y_1 - v_1y_3 \\ v_1y_2 - v_2y_1 \end{bmatrix} \quad \text{for } v = (v_1, v_2, v_3),\ y = (y_1, y_2, y_3),$$

a linear transformation?

a.) Need to find $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$. If $\begin{bmatrix}1\\0\end{bmatrix} = x\begin{bmatrix}3\\1\end{bmatrix} + y\begin{bmatrix}1\\2\end{bmatrix}$, we want to solve $\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$. Then $\begin{bmatrix}3 & 1\\1 & 2\end{bmatrix}^{-1} = \frac{1}{5}\begin{bmatrix}2 & -1\\-1 & 3\end{bmatrix}$

So $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3 & 1\\1 & 2\end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2/5\\-1/5\end{bmatrix}$. So $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \frac{2}{5}\begin{bmatrix}9\\3\end{bmatrix} - \frac{1}{5}\begin{bmatrix}2\\4\end{bmatrix}$ $= \begin{bmatrix}16/5\\2/5\end{bmatrix}$. Similarly $\begin{bmatrix}3 & 1\\1 & 2\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1/5\\3/5\end{bmatrix}$ so $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) =$ $-\frac{1}{5}\begin{bmatrix}9\\3\end{bmatrix} + \frac{3}{5}\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}-3/5\\9/5\end{bmatrix}$. So $[T] = \boxed{\begin{bmatrix}16/5 & -3/5\\2/5 & 9/5\end{bmatrix}}$.

b.) We need to check that $F(\vec{x}+\vec{y}) = F(\vec{x}) + F(\vec{y})$ and $F(k\vec{x}) = kF(\vec{x})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^3$ and $k \in \mathbb{R}$. This follows if the cross product is liner, because then we have $F(\vec{x}+\vec{y}) = \vec{v} \times (\vec{x}+\vec{y}) + A(\vec{x}+\vec{y}) = (\vec{v} \times \vec{x} + A\vec{x}) + (\vec{v} \times \vec{y} + A\vec{y})$ $= F(\vec{x}) + F(\vec{y})$ and $F(k\vec{x}) = \vec{v} \times (k\vec{x}) + A(k\vec{x})$ $= k(\vec{v} \times \vec{x}) + k(A\vec{x}) = kF(\vec{x})$. To check $\vec{v} \times \vec{x}$ is linear, write $\vec{x} = (x_1, x_2, x_3)$. Then

$$\vec{v} \times (\vec{x}+\vec{y}) = \begin{bmatrix} v_2(x_3+y_3) - v_3(x_2+y_2) \\ v_3(x_1+y_1) - v_1(x_3+y_3) \\ v_1(x_2+y_2) - v_2(x_1+y_1) \end{bmatrix} = \begin{bmatrix} v_2x_3 - v_3x_2 \\ v_3x_1 - v_1x_3 \\ v_1x_2 - v_2x_1 \end{bmatrix}$$

$$+ \begin{bmatrix} v_2y_3 - v_3y_2 \\ v_3y_1 - v_1y_3 \\ v_1y_2 - v_2y_3 \end{bmatrix} = \vec{v}\times\vec{x} + \vec{v}\times\vec{y}.$$ We also have

$$\vec{v} \times (k\vec{x}) = \begin{bmatrix} v_2(kx_3) - v_3(kx_2) \\ v_3(kx_1) - v_1(kx_3) \\ v_1(kx_2) - v_2(kx_1) \end{bmatrix} = k \begin{bmatrix} v_2x_3 - v_3x_2 \\ v_3x_1 - v_1x_3 \\ v_1x_2 - v_2x_1 \end{bmatrix}$$

$= k(\vec{v}\times\vec{x})$. So the cross product is linear and we are done.

Problem 4. *(25 points)*

(a) *(15 points)* Let A be a 1000×1000 matrix with rows v_j, $1 \leq j \leq 1000$ such that

$$v_{10} = -7v_{501} + 36v_{734}.$$

Is A invertible? Justify your answer.

(b) *(10 points)* Let A be an $n \times n$ matrix that is not the 0 matrix such that

$$A^2 = 0.$$

Show that A cannot be invertible. In the case of $n = 2$ give an example of such a matrix.

a.) Doing the sequence of row operations $36V_{734} \to V_{734}$, $-7V_{501} + V_{734} \to V_{734}$, $-V_{10} + V_{734} \to V_{734}$ produces a 0 row. So rref(A) has a 0 row, so rref$(A) \neq I$ and A is not invertible.

b.) If A was invertible, then $A^{-1}(A^2) = A^{-1}(0)$ $\Rightarrow$ $A = 0$, a contradiction. So A is not invertible.

For the example, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.