Math 33A - Midterm 2

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Discussion session: 3 A

Problems	Points	Score
1	35	35
2	30	30
3	25	23
4	10	1
Total	100	89

Problem 1. *(35 points)* Let Z be the subspace of $\mathbb{R}^3$ such that the set:

$$\mathcal{B} = \left\{ v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\},$$

is a basis for Z.

(a) *(15 points)* Construct an orthonormal basis of Z from the basis $\mathcal{B}$. *Hint: Apply the Gram-Schmidt process to the vectors of $\mathcal{B}$.*

(b) *(20 points)* Given that the vector

$$w = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

belongs to Z, find the $\mathcal{B}$-coordinates of w.

a) $\sqrt{3}$ $\quad u_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

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$u_2 = v_2 - \langle v_2, u_1 \rangle u_1 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - 12/\sqrt{3} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} / \sqrt{4+4} =$

$\begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

b) $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

$2 = c_1 + 2c_2$

$1 = c_1 + 4c_2$

$0 = c_1 + 6c_2 \qquad c_1 = 3$

$\begin{bmatrix} 3 \\ -1/2 \end{bmatrix}$

$2 = -4c_2$

$c_2 = -1/2$

Problem 2. *(30 points)*

(a) *(20 points)* Consider the following subspace V of $\mathbb{R}^3$:

$$V = \{(x, y, z) \in \mathbb{R}^3 : x + 3y + 5z = 0\}.$$

Find a basis of V.

Hint: There are many ways to do this problem. An easy one is to write V as the kernel of a matrix.

(b) *(10 points)* Let V be as in part (a), and let W be the following subspace of $\mathbb{R}^3$:

$$W = \{(x, y, z) \in \mathbb{R}^3 : x - 3y + 5z = 0\}.$$

Is the union of V and W, which is the set

$$V \cup W = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \in V \text{ or } (x, y, z) \in W\},$$

a subspace of $\mathbb{R}^3$? Justify your answer.

a) $V = \ker(\begin{bmatrix} 1 & 3 & 5 \end{bmatrix})$ $\quad \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0}_{\mathbb{R}^3}$

take $c_1 = y$ and $c_2 = z$, $\quad x = -3y - 5z$

then $c_1\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$ spans V, with 2 Lin. Independent vectors

$\therefore$ $\boxed{\beta_V = \left\{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}\right\}}$

b) $\beta_W = \left\{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}\right\}$

take v_1 as $c_1\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$, in V and

w_1 as $d_1\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + d_2\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$ in W.

$v_1 + w_1 = c_1\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + d_1\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + (c_2 + d_2)\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$.

for example, $c_1 = 1, d_1 = 1, c_2 = d_2 = 0$, $v_1 + w_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$, not in V or W

Seeing as, when both c_1 and d_1 are non-zero, the result is neither in V nor W, $V \cup W$ is not closed under addition, so it is not a subspace of $\mathbb{R}^3$.

1 1   1 1
1 2   1 2
2 3
3 5

Problem 3. *(25 points)*

(a) *(15 points)* Assume that A and B are two invertible $n \times n$ matrices. Describe the image (range) of AB:

$$Im(AB).$$

Justify your answer.

(b) *(10 points)* Can there exist an invertible 2×2 matrix C such that

$$ker(C) = span\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}?$$

Justify your answer.

Hint: Use the rank-nullity Theorem.

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a) Because A and B are both invertible, $Im(A) = Im(B) = \mathbb{R}^n$. Viewing AB as a composition of linear transformation functions, $\mathbb{R}^n \to \mathbb{R}^n \to \mathbb{R}^n$, it is clear that the set of all outputs from ABx will be the same as the set of all outputs from Bx_A, where x_A is $\mathbb{R}^n$, the set of all outputs from Ax, therefore, $Im(AB)$ is $\mathbb{R}^n$ ✓

~~very~~ very informal

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b) For any invertible matrix, the rank of A = n, ∴ by rank-nullity thrm, $\dim(\ker(A)) = 0$, the basis $\begin{bmatrix}1\\1\end{bmatrix}$ is of dimension 1, therefore it can't be the basis of a kernel of an invertible matrix of any size.

Problem 4. *(10 points)* Let V and W be two subspaces of $\mathbb{R}^n$. Assume that

$$V \subseteq W \text{ and that } dim(V) = dim(W).$$

Show that

given $V = W$.

$V \subseteq W$ is ~~known~~, only need $W \subseteq V$ to say ~~W=V~~ $V = W$

Let $B_V = \{v_1 \dots v_m\}$ be a basis for V

and $B_W = \{v_1 \dots v_m, w_1 \dots w_k\}$ be a basis for W. because $V \subseteq W$, unique to W

For $dim(V) = dim(W)$ and $V \subseteq W$, that is, W must consist of all elements of V ($span\{B_V\}$), there can be no additional linearly independent vectors $w_1 \dots w_k$ in W, otherwise $dim(W) > dim(V)$ or $V \subsetneq W$, and B_W must also be a basis for V. $\therefore$ $W \subseteq V$ and $V \subseteq W$, so $V = W$

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